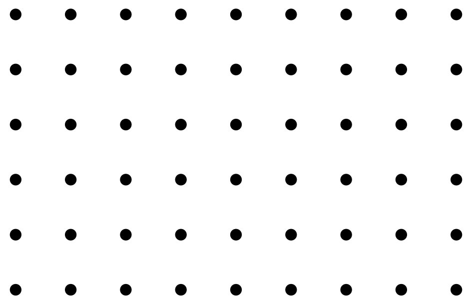
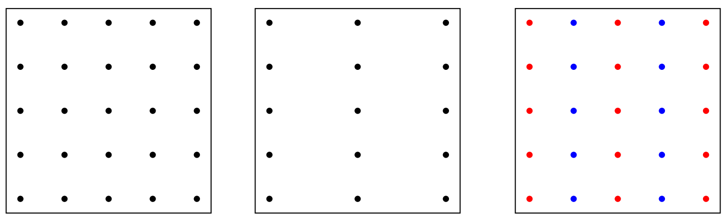
2. Lattices and quaternion algebras

A *lattice* in the plane is an infinitely repeating grid of points in the plane containing the origin.



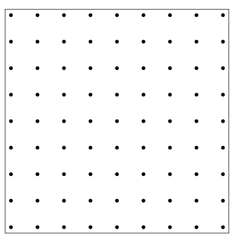
Now let us consider how two lattices can be related. If Λ_1, Λ_2 are lattices satisfying $\Lambda_2 \subseteq \Lambda_1$, then Λ_1 is a union of finitely many translates of Λ_1. The number of translates is called the *index* of Λ_2 in Λ_1, denoted $[\Lambda_1 : \Lambda_2]$.

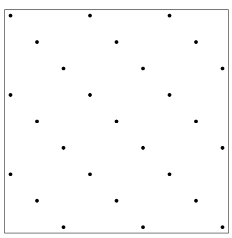
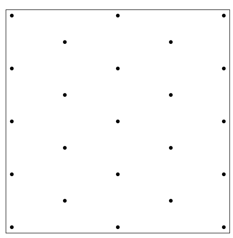
Example 2.1. Here is an example where $[\Lambda_1 : \Lambda_2] = 2$.



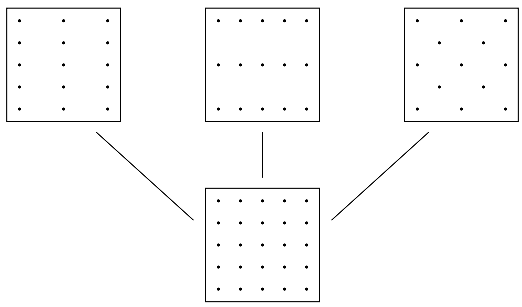
Λ_1 $\qquad$ Λ_2 $\qquad$ translates of Λ_2 in Λ_1

Exercise 2.2. Compute $[\Lambda_1 : \Lambda_2]$ where:

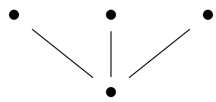
$\Lambda_1 =$ 

$\Lambda_2 =$  or $\Lambda_2 =$  .

For any lattice Λ_1, there are exactly three lattices Λ_2 satisfying $\Lambda_2 \subset \Lambda_1$ and $[\Lambda_1 : \Lambda_2] = 2$.



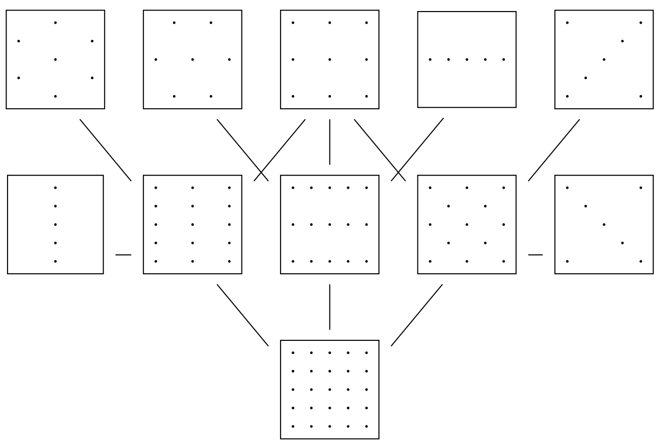
We can summarize this information in a graph.



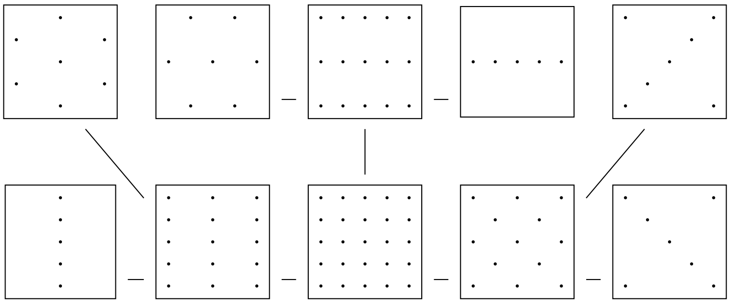
Now let us construct a graph as follows:

- Each lattice in the plane is a vertex of the graph.
- Draw an edge between the lattice Λ_1 and the lattice Λ_2 if $\Lambda_2 \subset \Lambda_1$ and $[\Lambda_1 : \Lambda_2] = 2$.

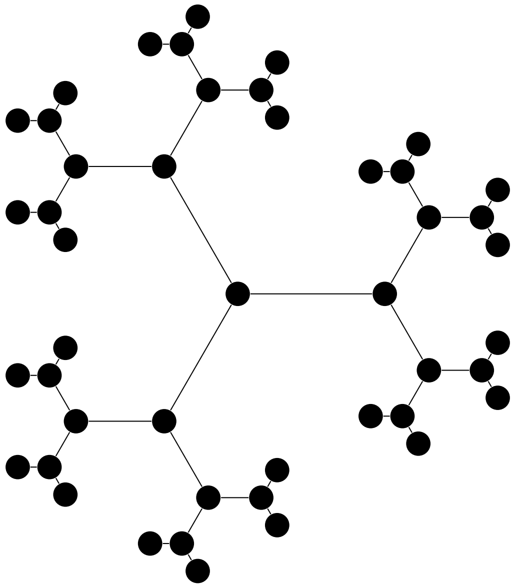
We have already drawn a small piece of this graph. Here is a larger piece:



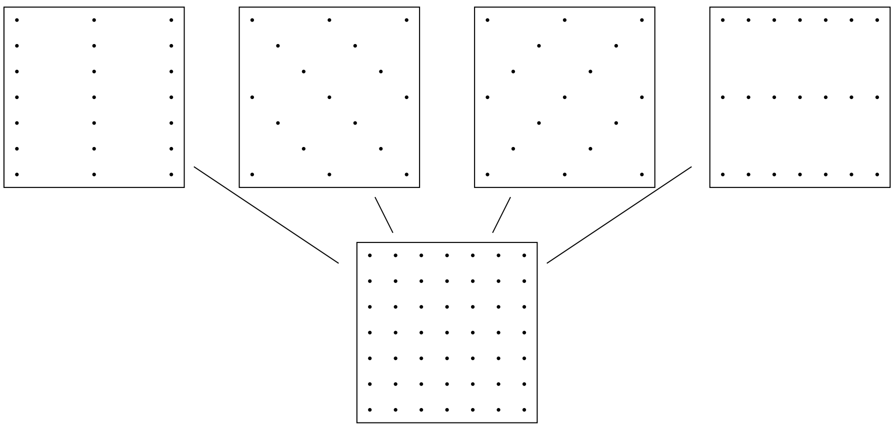
We can simplify the diagram a bit by declaring two lattices to be equivalent if they are related by scaling. In particular, the bottom lattice is equivalent to the top middle lattice. After identifying equivalent lattices, the piece of the graph looks like this.



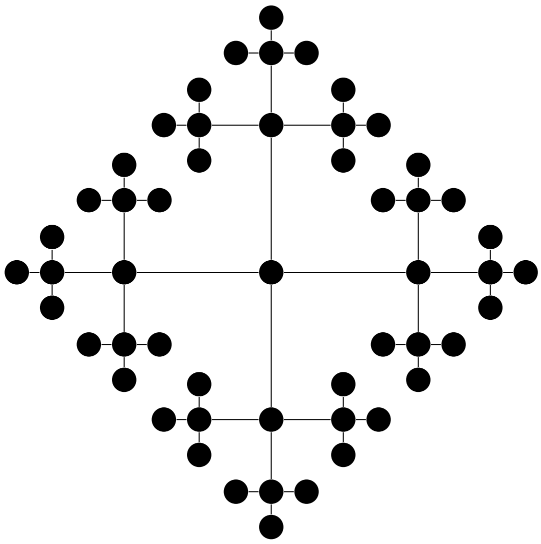
The equivalence classes of lattices form a tree, called the *Bruhat–Tits tree*.



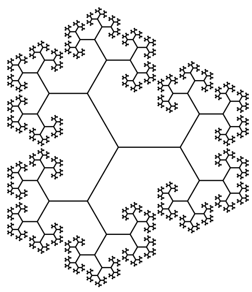
Now suppose that instead we draw an edge between lattices Λ_1, Λ_2 satisfying $[\Lambda_1 : \Lambda_2] = 3$.



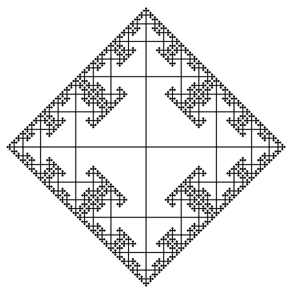
The graph is then an infinite 4-regular tree:



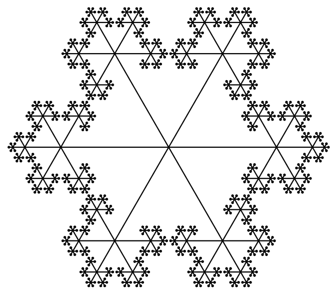
In general, for any prime p, if we draw an edge between pairs of lattices satisfying $[\Lambda_1 : \Lambda_2] = p$, we will get a $p+1$-regular tree.



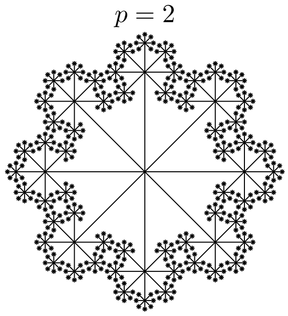
$p = 2$



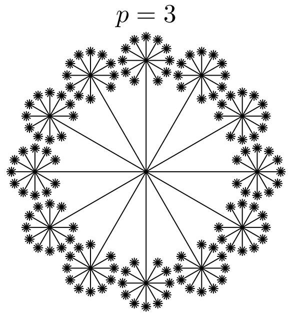
$p = 3$



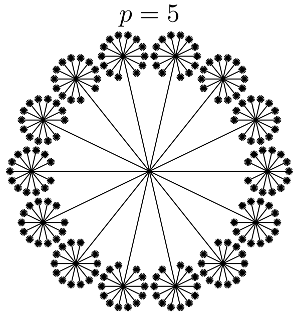
$p = 5$



$p = 7$

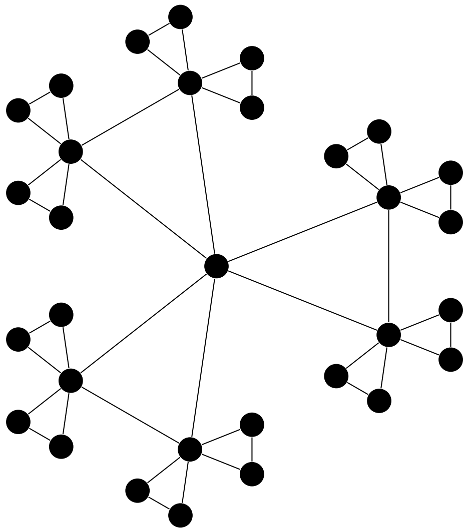


$p = 11$



$p = 13$

On the other hand, if we draw an edge between pairs of lattices satisfying $[\Lambda_1 : \Lambda_2] = n$, where n is not prime, then we generally don't get a tree. For example, when $n = 4$, the graph looks like this:



Now let's consider two different ways that we could generalize this setup:

(1) Look at higher-dimensional lattices.
(2) Look at lattices with symmetry.

So, for any n we could consider constructing a graph as follows:

(1) Vertices are n-dimensional lattices up to scaling.
(2) Draw an edge between two vertices if there are representatives Λ_1, Λ_2 such that $\Lambda_2 \subset \Lambda_1$ and $[\Lambda_1 : \Lambda_2] = 2$.

The resulting graph is called a *building*. When $n = 3$, the building looks something like this:

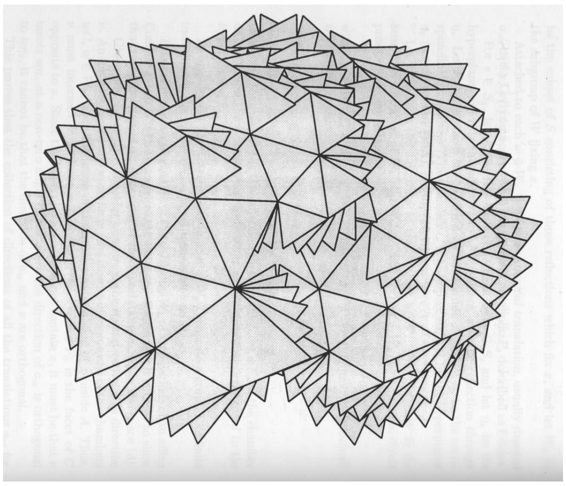
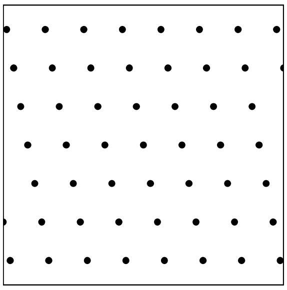


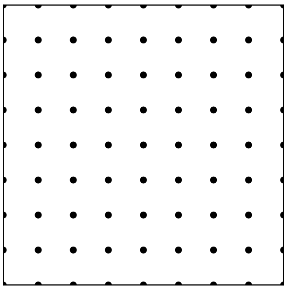
Image credit: P. Garrett, *Buildings and Classical Groups*

Unfortunately, I don't have more to say about buildings. Our construction of Ramanujan graphs will only use the Bruhat–Tits trees.

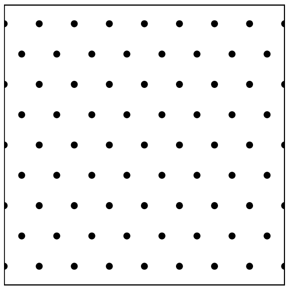
Now let's look at symmetries of lattices. Any lattice has twofold rotational symmetry. Some lattices in the plane have fourfold or sixfold rotational symmetry.



Twofold symmetry

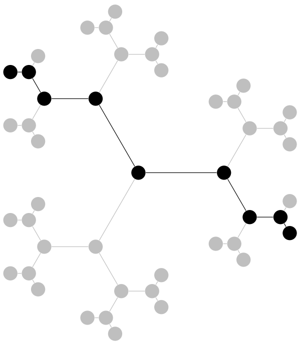


Fourfold symmetry

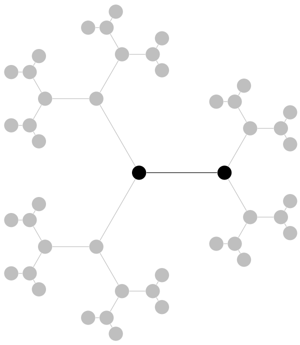


Sixfold symmetry

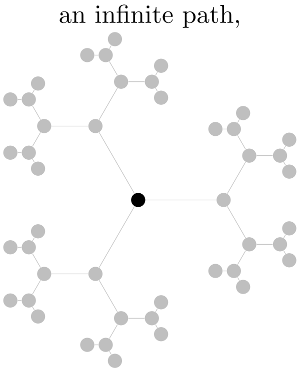
Most lattices only have twofold rotational symmetry. On the Bruhat-Tits tree, the set of points that correspond to lattices with fourfold or sixfold symmetry is either:



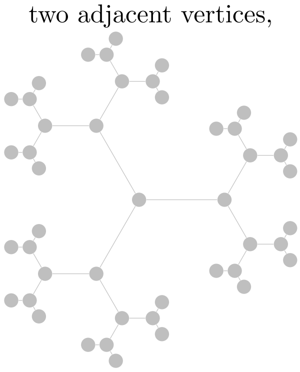
an infinite path,



two adjacent vertices,



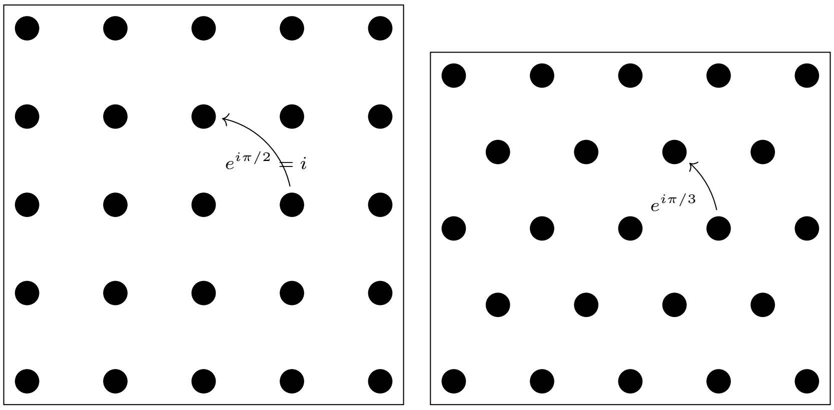
a single vertex, or



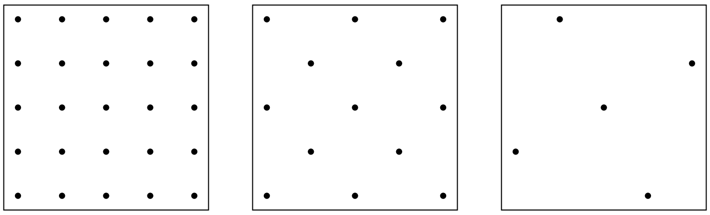
empty.

So we do not get very interesting graphs this way. However, we can get interesting graphs if we look at symmetrical lattices in higher dimensions. In particular, we will construct Ramanujan graphs using symmetrical lattices in the four dimensions.

It will be useful to think of symmetries in the following way. Rotations in the plane have a nice interpretation in terms of complex numbers. In the complex plane, multiplication by $e^{i\theta}$ corresponds to rotation by θ. So a lattice in the complex plane has fourfold rotational symmetry if it is invariant under multiplication by $e^{i\pi/2} = i$, and sixfold rotational symmetry if it is invariant under multiplication by $e^{i\pi/3}$.

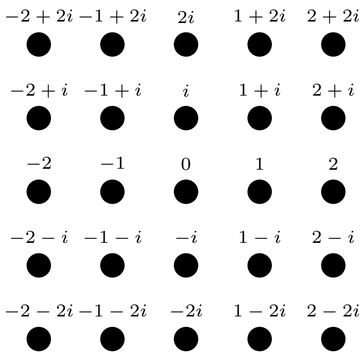


Another way of thinking about this is the following. Suppose that the lattice Λ has fourfold rotational symmetry. Then $i\Lambda = \Lambda$. Note that $(1+i)\Lambda \subset \Lambda$, since the sum of two elements of Λ is also in Λ. Similarly, $(a+bi)\Lambda \subset \Lambda$ for all $a, b \in \mathbb{Z}$.



Λ $\quad$ $(1+i)\Lambda$ $\quad$ $(2+i)\Lambda$

Define $\mathbb{Z}[i] := \{a + bi | a, b \in \mathbb{Z}\}$.



A lattice Λ as fourfold rotational symmetry if and only if

$$\{z \in \mathbb{C} | z\Lambda \subseteq \Lambda\} = \mathbb{Z}[i] .$$

If this condition is satisfied, we say that Λ is a $\mathbb{Z}[i]$-*ideal.*

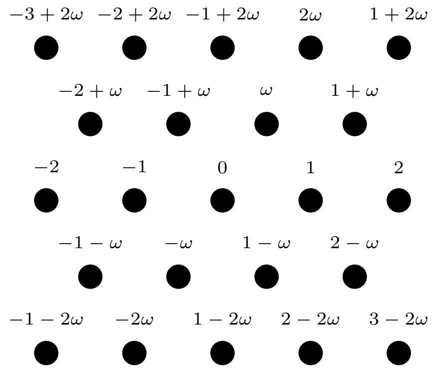
The lattice $\mathbb{Z}[i]$ is an example of an *order*:

Definition 2.3. An *order* in $\mathbb{C}$ is a lattice $\mathcal{O} \subset \mathbb{C}$ such that:

(1) $1 \in \mathcal{O}$
(2) For all $z_1, z_2 \in \mathcal{O}$, $z_1 z_2 \in \mathcal{O}$.

Similarly, we can consider the lattice

$$\mathbb{Z}[\omega] := \{a + b\omega | a, b \in \mathbb{Z}\}\,, \text{ where } \omega = e^{\pi i/3} = \frac{1}{2} + \frac{\sqrt{3}}{2} i\,.$$



A lattice Λ has sixfold rotational symmetry if and only if

$$\{z \in \mathbb{C} | z\Lambda \subseteq \Lambda\} = \mathbb{Z}[\omega]\,.$$

Furthermore, $\mathbb{Z}[\omega]$ is an order.

For any lattice $\Lambda \subset \mathbb{C}$, define

$$\mathrm{End}(\Lambda) := \{z \in \mathbb{C} | z\Lambda \subseteq \Lambda\}\;.$$

For most lattices Λ, $\mathrm{End}(\Lambda)$ will just consist of integers. However, we have seen that $\mathrm{End}(\Lambda)$ can be an order such as $\mathbb{Z}[i]$ or $\mathbb{Z}[\omega]$.

Our construction of Ramanujan graphs will involve symmetrical four-dimensional lattices. We need four-dimensional analogue of the complex numbers. These are the *quaternions*.

Definition 2.4. The *quaternions* are the set

$$\mathbb{H} := \{a + bi + cj + dk | a, b, c, d \in \mathbb{R}\}$$

We define multiplication of two quaternions by the rules

$$ij = -ji = k, \quad jk = -kj = i, \quad ki = -ik = j,$$
$$i^2 = -1, \quad j^2 = -1, \quad k^2 = -1\,.$$

Unlike the case of the complex numbers, multiplication is not commutative.

We define orders and ideals similarly to the complex case.

Definition 2.5. An *order* in $\mathbb{H}$ is a lattice $\mathcal{O} \subset \mathbb{H}$ such that

(1) $1 \in \mathcal{O}$.
(2) For all $z_1, z_2 \in \mathcal{O}$, $z_1 z_2 \in \mathcal{O}$.

Definition 2.6. A *left $\mathcal{O}$-ideal* is a lattice $\Lambda \subset \mathbb{H}$ such that

$$\{x \in \mathbb{H} | x\Lambda \subseteq \Lambda\} = \mathcal{O}\,.$$

Exercise 2.7. Which of these subsets of $\mathbb{H}$ are orders in $\mathbb{H}$?

$$\mathbb{Z}$$
$$\{a + bi + cj + dk | a, b, c, d \in \mathbb{Z}\}$$
$$\{a + bi + cj + dk | a, b, c, d \in 2\mathbb{Z}\}$$

$$\{a + bi + cj + dk | a \in \mathbb{Z},\ b, c, d \in 2\mathbb{Z}\}$$
$$\{a + bi + cj + dk | a, b, c, d \in \frac{1}{2}\mathbb{Z}\}$$

Example 2.8. The four-dimensional lattice $\mathcal{O}$ consisting of integer combinations of

$$1, \quad \frac{i - \sqrt{3}k}{2}, \quad i - \sqrt{3}j, \quad \frac{1 + 3i + \sqrt{3}j + \sqrt{3}k}{2}$$

is an order. To verify this, we need to check that any product of two of the four generators above is again in $\mathcal{O}$. For example, we can check that

$$\begin{aligned}
&\left(\frac{i - \sqrt{3}k}{2}\right)\left(i - \sqrt{3}j\right) \\
=&\frac{1}{2}\left(i \cdot i - i \cdot \sqrt{3}j - \sqrt{3}k \cdot i + \sqrt{3}k \cdot \sqrt{3}j\right) \\
=&\frac{1}{2}\left(-1 - \sqrt{3}k - \sqrt{3}j - 3i\right) \\
=&-\frac{1 + 3i + \sqrt{3}j + \sqrt{3}k}{2}.
\end{aligned}$$

We leave the remainder of the verification to the reader.