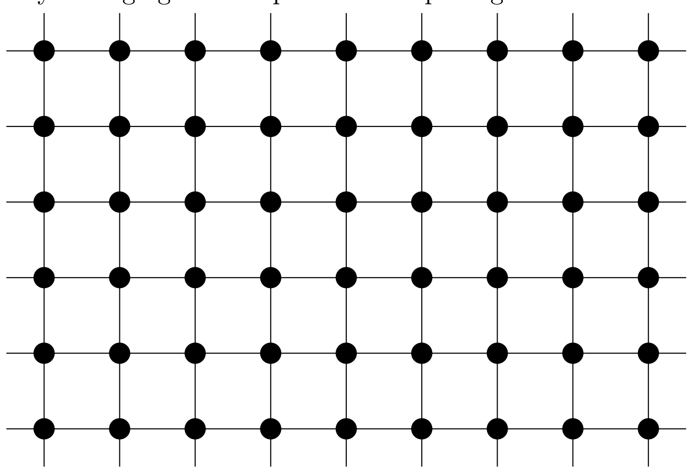
1. Ramanujan graphs

Suppose we are designing a computer network. We would like for there to be a short path from any computer to any other computer. In principle, we could just connect every pair of computers, but that would be expensive. So let's say that we can only afford to connect each computer to, say, four others. For simplicity, let's assume that it doesn't matter where the computers are physically located, only which ones are connected.

We could try arranging the computers in a square grid.



Question 1.1.

(1) How would you quantify how good a network is?
(2) How would you design a network that is better than the one above?

Definition 1.2. Let k be a positive integer. A graph is *k-regular* if there are k edges connected to each vertex.

There are many ways that we could quantify how well-connected a graph is. A fairly obvious one is the diameter of the graph.

Definition 1.3. The *distance* between two points in a graph is the length of the shortest path between them.

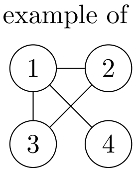
The *diameter* of a graph is the maximum distance between any pair of points in the graph.

By this measure, a square grid is certainly not optimal. We can decrease the diameter by removing some edges between nearby vertices and replacing them with edges between far away vertices.

Another, less obvious, way of measuring the connectedness of a graph is to look at the eigenvalues of its adjacency matrix.

Definition 1.4. Let G be a graph. The *adjacency matrix* A of G is the matrix with $A_{ij} = 1$ if vertices i and j are connected, and $A_{ij} = 0$ otherwise.

Example 1.5. Here is an example of a graph and its adjacency matrix.



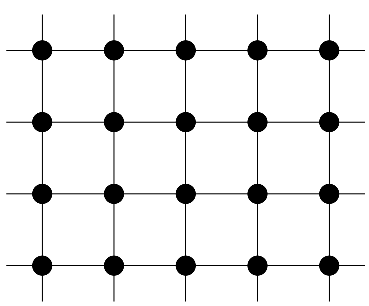
$$\begin{pmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

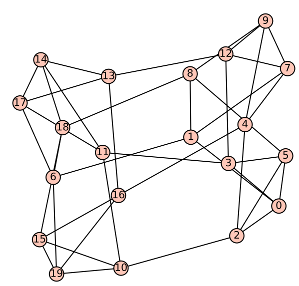
The matrix A records the number of paths of length 1 between any two vertices. To compute the number of paths of some other length, we take powers of A:

Lemma 1.6. *Let G be a graph, and let A be its adjacency matrix. For any vertices i, j of G and any nonnegative integer n, the number of paths of length n from i to j is A^n.*

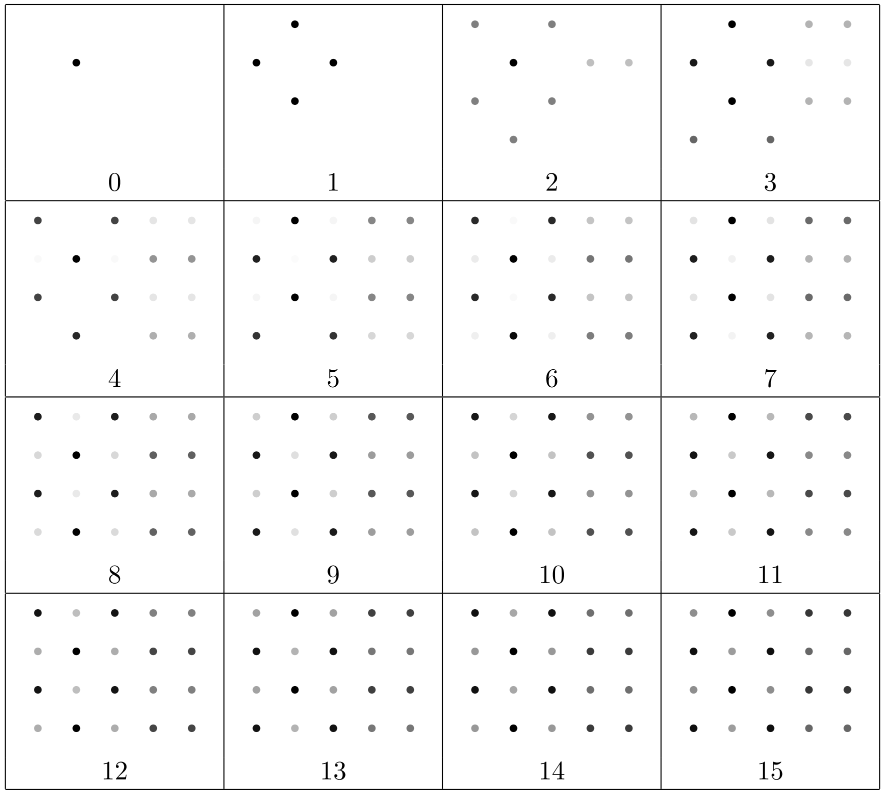
Rather than just looking at the shortest path from vertex i to vertex j, we can count the number of paths from i to j of a given distance. If we need to send a lot of data from vertex i to vertex j, then we may want to use more than one path i to j. Having many short paths from i to j is also useful in case one of the connections breaks.

Now let's look at two graphs in particular.



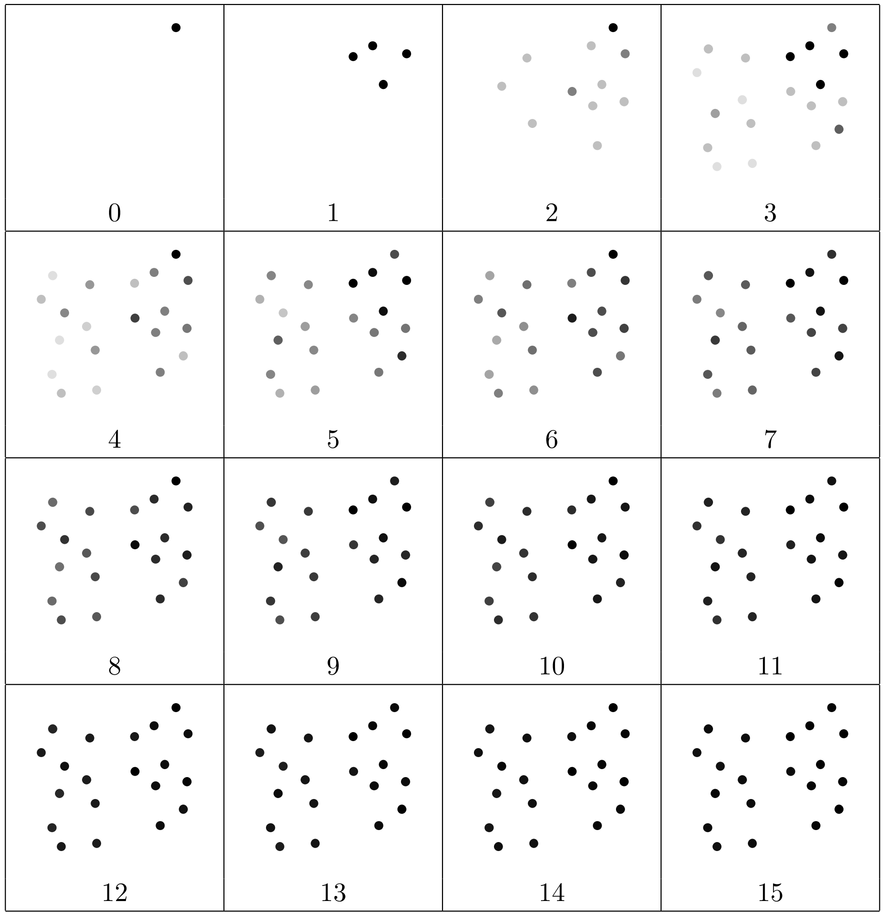


For each graph, we will choose a vertex and plot a diagram showing the number of paths of a given length from that vertex to each other vertex. We start with the graph on the left. The darkness of the dot indicates the number of paths, relative to the maximum.



There are a decent number of paths of length 15 from the starting node to any other node, but there are still noticeably more paths to some nodes than others.

By contrast, the plot for the graph on the right looks like this:



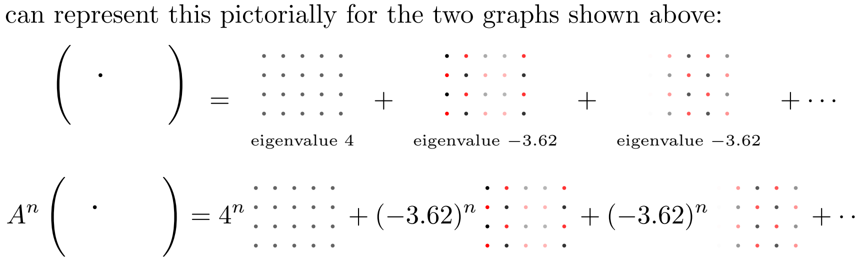
The number of paths becomes uniform a lot quicker for this graph. The difference can be explained in terms of the eigenvalues of the adjacency matrices of these graphs. Let i, be the starting vertex, and let v be the corresponding vector, i.e. the ith entry of v is 1 and the other entries of v are zero. For any vertex j and nonnegative integer d, the number of paths of length d from i to j is the jth entry of $A^d v$. We can write

$$v = e_1 + \cdots + e_n \,,$$

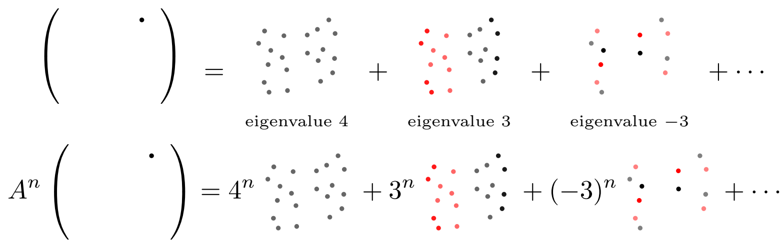
where $e_1, \ldots, e_n$ are eigenvectors of A with respective eigenvalues $\lambda_1, \ldots, \lambda_n$. Then for any $d \geq 0$,

$$A^d v = \lambda_1^d e_1 + \cdots + \lambda_n^d e_n \,.$$

We can represent this pictorially for the two graphs shown above:



$$4^{15} \approx 1.1 \times 10^9, \quad (-3.62)^{15} \approx -2.4 \times 10^8$$



$$4^{15} \approx 1.1 \times 10^9, \quad 3^{15} \approx 1.4 \times 10^7, \quad (-3)^{15} \approx -1.4 \times 10^7$$

Since both graphs are 4-regular, they each have 4 as an eigenvalue. The next-largest eigenvalues of the first graph are about -3.62, while the next-largest eigenvalues of the second graph are ± 3. The numbers of paths in the second graph are more uniform because the eigenvalues are smaller.

How small can the eigenvalues be?

Lemma 1.7. *A real symmetric matrix has real eigenvalues.*

Definition 1.8. Let G be a graph. We will write $\lambda_i(G)$ for the ith largest eigenvalue of the adjacency matrix of the graph G.

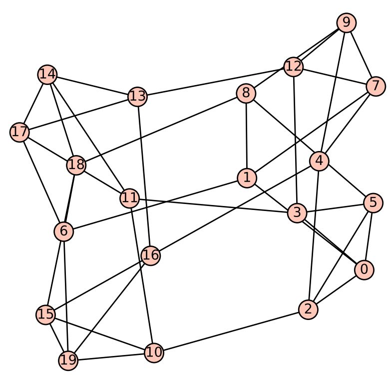
Proposition 1.9 ([Nil91])**.** *Let G be a connected k-regular graph. Suppose that G has two edges of distance at least $2m$ apart. Then*

$$\lambda_2(G) \geq 2\sqrt{k-1} - \frac{2\sqrt{k-1}-1}{m}.$$

In particular, this bound approaches $2\sqrt{k-1}$ as the number of vertices of G goes to infinity.

Definition 1.10. A connected k-regular graph is *Ramanujan* if $|\lambda_i(G)| \leq 2\sqrt{k-1}$ for $i > 1$.

Example 1.11. The following is a 4-regular Ramanujan graph with 20 vertices.



The graph was generated with the following Sage code:

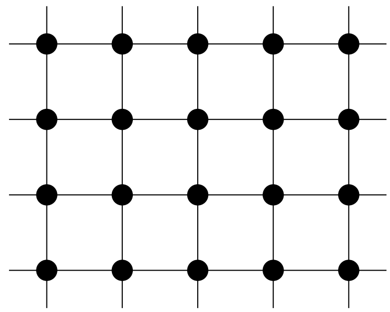
```
M = BrandtModule(2,175)
G = Graph(M.hecke_matrix(3),format='adjacency_matrix')
G.plot()
```

The following Sage code computes eigenvalues of the adjacency matrix of G:

```
sorted(G.adjacency_matrix().eigenvalues())
```

We find that the largest nontrivial eigenvalues of the adjacency matrix are ± 3, while the Ramanujan bound is $2\sqrt{3} \approx 3.46$.

Non-example 1.12. A 4×5 wrapping square grid is not Ramanujan.



The eigenvalues of its adjacency matrix are of the form $2\cos(2\pi\ell/4) + 2\cos(2\pi m/5)$ for integer ℓ, m. In particular, $2\cos\pi + 2\cos(4\pi/5) = -\frac{5+\sqrt{5}}{4} \approx -3.62$ is an eigenvalue.

Graphs with small eigenvalues tend to do well on other measures of connectedness, including the diameter.

Proposition 1.13 ([Sar90], Proposition 3.2.6)**.** *Let G be a k-regular graph with n vertices. Let $\mu = \max(|\lambda_2|, |\lambda_n|)$. Then the diameter of G is at most*

$$\frac{\log(2n)}{\log\left(\frac{k+\sqrt{k^2-\mu^2}}{\mu}\right)}.$$

In particular, if G is Ramanujan, then the diameter of G is at most

$$\frac{2\log(2n)}{\log(k-1)}.$$

In contrast, the diameter of a wrapping square grid like the one in Non-Example 1.12 grows as the square root of the number of vertices.

References

[Nil91] A. Nilli. On the second eigenvalue of a graph. *Discrete Math.*, 91(2):207–210, 1991.

[Sar90] P. Sarnak. *Some applications of modular forms*, volume 99 of *Cambridge Tracts in Mathematics*. Cambridge University Press, Cambridge, 1990.