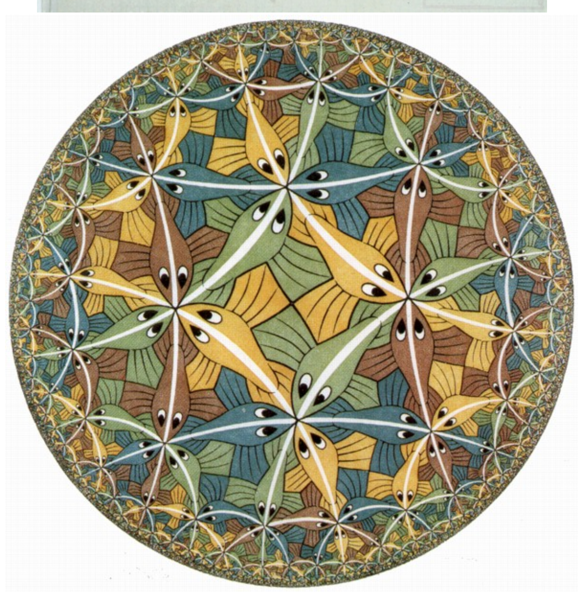
Kleinian Groups and Fractals homework - Day 2

1. Figure 18 in the notes shows how the corners of a tile lead to relations in the tiling group. Draw similar diagrams for the two tilings below.



2. Given four points $z_1, z_2, z_3, z_4 \in \hat{\mathbb{C}}$, the *cross ratio* $(z_1, z_2; z_3, z_4)$ is defined to be

$$\frac{(z_3 - z_1)(z_4 - z_2)}{(z_3 - z_2)(z_4 - z_1)}.$$

This seemingly random quantity turns out to be quite useful!

(a) Check that Möbius transformations preserve the cross ratio.

(b) Show that four points $z_1, z_2, z_3, z_4 \in \hat{\mathbb{C}}$ lie on a common circle or line if and only if $(z_1, z_2; z_3, z_4) \in \mathbb{R} \cup \{\infty\}$. (Hint: the argument of $\frac{z_3 - z_1}{z_3 - z_2}$ is the measure of angle $z_1 z_3 z_2$.) In particular, this means that Möbius transformations map circles and lines to circles and lines.

(c) Show that Möbius transformations preserve angles. (Hint: one way to do this is the following. Let C_1, C_2 be intersecting circles, let z_1 be a point on C_1, let z_2 be a point on C_2, and let z_3, z_4 be the intersection points of C_1 and C_2. Express the angle at which C_1 and C_2 intersect in terms of the cross ratio $(z_1, z_2; z_3, z_4)$.)

3. Write down an explicit set of generators for a Schottky group. In other words, choose four disjoint discs $D_a, D_{a^{-1}}, D_b, D_{b^{-1}} \subset \hat{\mathbb{C}}$ and write down transformations a, b such that a maps the exterior of $D_{a^{-1}}$ to D_a and b maps the exterior of $D_{b^{-1}}$ to D_b. (Hint: The map $z \mapsto 1/z$ interchanges the inside and outside of the unit disc centered at the origin. Try composing this map with translations.) If you enjoy programming, write a computer program to plot some orbits of the group generated by a and b.

4. If you have not seen Cayley graphs before, read the Wikipedia article: `https://en.wikipedia.org/wiki/Cayley_graph`.

(a) In the top left right corner of the Wikipedia article is a Cayley graph for a free group on two generators. Do you see a resemblance between this graph and Schottky tilings?

(b) Draw a Cayley graph for the symmetry group of one of Escher's drawings.