Kleinian Groups and Fractals homework - Day 1

Exercises 1–4 will be useful later in the class. Exercises 5–8 are just for fun.

1. Verify that the set of Möbius transformations forms a group under composition.

2. Let $\mathrm{GL}_2(\mathbb{C})$ be the set of invertible 2×2 complex matrices. These form a group under multiplication. Define a map from $\mathrm{GL}_2(\mathbb{C})$ to the group of Möbius transformations that sends the matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ to the transformation $z \mapsto \frac{az+b}{cz+d}$.

   (a) Verify that the formulas for composing Möbius transformations are the same as the formulas for multiplying 2×2 matrices. In other words, the above map is a homomorphism.

   (b) Also convince yourself that the above map is surjective and that its kernel (the set of elements mapping to the identity) is the set of scalar multiples of the identity matrix. Conclude that the group of Möbius transformations is $\mathrm{GL}_2(\mathbb{C})/\mathbb{C}^\times$. This group is often denoted $\mathrm{PGL}_2(\mathbb{C})$. (The P stands for "projective".)

   If you are not so familiar with group theory and find this exercise confusing, it may help to read the first two chapters of `https://pages.mtu.edu/~kreher/ABOUTME/syllabus/GTN.pdf`. In particular, part (b) uses a result in group theory called the First Isomorphism Theorem, which is Theorem 2.6.1 in the above notes.

3. (a) Let a be a rotation in the plane about a point P, and let b be any symmetry of the plane. Convince yourself that bab^{-1} is a rotation about bP, with the same angle as a.

   (b) The *trace* of a matrix (sometimes denoted tr) is defined to be the sum of its diagonal entries. Let A and B be 2×2 matrices, and assume that B is invertible. Show that $\operatorname{tr} A = \operatorname{tr} BAB^{-1}$. (Hint: to simplify the algebra, first show that for any 2×2 matrices C and D, $\operatorname{tr} CD = \operatorname{tr} DC$, then let $C = AB^{-1}$, $D = B$.)

4. Find a Möbius transformation that takes the interior of the circle of radius 1 centered at $z = -3/2$ to the exterior of the circle of radius 1 centered at $z = 3/2$. (Hint: use the fact that $z \mapsto -1/z$ interchanges the interior and exterior of the circle of radius 1 centered at $z = 0$.)

5. Given a point $z = x + iy \in \mathbb{C}$, find the coordinates of its stereographic projection. Assume that the sphere has equation $x^2+y^2+(z-1/2)^2 = 1/4$, so that the sphere is tangent to the complex plane at its south pole $(0,0,0)$ and its north pole has coordinates $(0,0,1)$.

6. Let SU(1, 1) be the set of matrices of the form

$$\begin{pmatrix} a & b \\ \bar{b} & \bar{a} \end{pmatrix}, \quad a, b \in \mathbb{C}, \quad |a|^2 - |b|^2 = 1\,.$$

   (a) Verify that SU(1, 1) is a subgroup of $\mathrm{GL}_2(\mathbb{C})$.

   (b) By exercise 2, each element of $\mathrm{GL}_2(\mathbb{C})$ determines a Möbius transformation. Verify that any element of SU(1, 1) maps the interior of the unit circle to itself.

   (c) Verify that any Möbius transformation arising from an element of SU(1, 1) commutes with the map $z \mapsto 1/\bar{z}$.

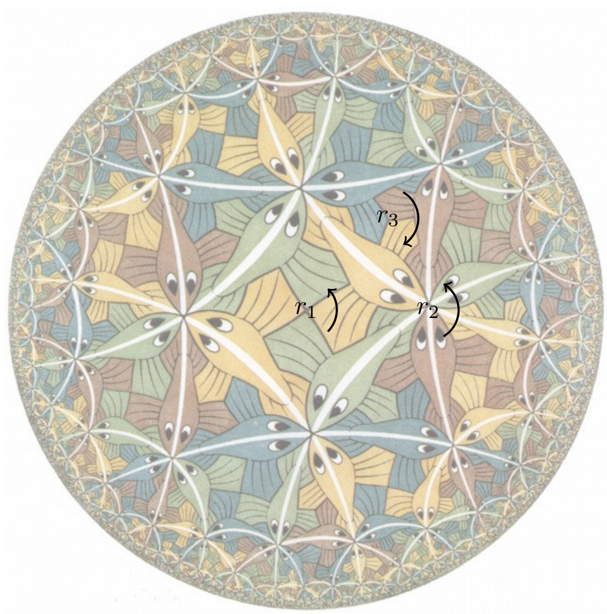
   (d) Verify that if $A, B \in \mathrm{SU}(1, 1)$ represent the same Möbius transformation, then $A = \pm B$.

7. In class, it was claimed that the transformation

$$z \mapsto \frac{(1 - e^{i\pi/4})^{-1} z - 2^{-1/4} i}{2^{-1/4} i z + (1 - e^{-i\pi/4})^{-1}}\,.$$

   is a symmetry of Escher's *Circle Limit III* drawing. In this exercise, we will justify this claim.

   (a) Consider the rotations r_1, r_2, r_3 depicted in the picture below. Convince yourself that $r_3 = r_1 r_2$. (Reminder: $r_1 r_2$ means "do r_2 first, then r_1", since $(r_1 r_2)(z)$ is defined to be $r_1(r_2(z))$.)



(b) Observe that the matrix $\pm\begin{pmatrix} e^{i\theta/2} & 0 \\ 0 & e^{-i\theta/2} \end{pmatrix} \in \mathrm{SU}(1,1)$ represents a counterclockwise rotation about the origin by angle θ. Based on exercise 3, convince yourself that a matrix representing a "rotation" by θ should have trace $\pm(e^{i\theta/2} + e^{-i\theta/2}) = \pm 2\cos\theta/2$.

(c) The transformations r_1, r_2, r_3 are "rotations" by $\pi/2$, $2\pi/3$, $-2\pi/3$, respectively. If A_1, A_2, A_3 are matrices corresponding to r_1, r_2, r_3, then we should have

$$A_3 = \pm A_2 A_1$$

$$A_1 = \pm\begin{pmatrix} e^{i\pi/4} & 0 \\ 0 & e^{-i\pi/4} \end{pmatrix}$$

$$\operatorname{tr} A_2 = \pm 2\cos\pi/3 = \pm 1$$

$$\operatorname{tr} A_3 = \pm 2\cos -\pi/3 = \pm 1\,.$$

We are free to multiply each matrix by -1, so we can fix most of the signs as follows:

$$A_3 = A_2 A_1$$

$$A_1 = \begin{pmatrix} e^{i\pi/4} & 0 \\ 0 & e^{-i\pi/4} \end{pmatrix}$$

$$\operatorname{tr} A_2 = 1$$

$$\operatorname{tr} A_3 = \operatorname{tr} A_2 A_1 = \pm 1\,.$$

Let $A_2 = \begin{pmatrix} a & b \\ \overline{b} & \overline{a} \end{pmatrix}$; then the above equations imply

$$a + \overline{a} = 1$$

$$e^{i\pi/4} a + e^{-i\pi/4}\overline{a} = \pm 1\,.$$

Solve for a. Check that only one of the two solutions satisfies $|a| \geq 1$ (which is necessary for $|a|^2 - |b|^2 = 1$).

(d) Check that if the transformation $z \mapsto \frac{az+b}{\overline{b}z+\overline{a}}$ fixes a point on the positive real axis other than 1, then $\operatorname{Re} b = 0$, and $\operatorname{Im} a$ and $\operatorname{Im} b$ have opposite signs. Use this information to solve for b, given the value of a found in part 7c.

8. Write a computer program that makes symmetric drawings. (If you need some inspiration, try searching for "p5.js tessellation".)