1. Introduction

1.1. **Cohomology of algebraic varieties.** Let X be a proper smooth algebraic variety over a field K. One can define various cohomology groups:

- For any embedding $K \hookrightarrow \mathbb{C}$, the Betti (singular) cohomology $H^*_B(X(\mathbb{C}), \mathbb{Z})$, an abelian group.
- The de Rham cohomology $H^*_{\mathrm{dR}}(X/K)$, a filtered K-vector space.
- For any prime ℓ, the ℓ-adic étale cohomology $H^*_{\text{ét}}(X_{K^{\mathrm{sep}}}, \mathbb{Z}_\ell)$, a $\mathbb{Z}_\ell$-module with $G_K := \mathrm{Gal}(K^{\mathrm{sep}}/K)$-action.
- If K has characteristic $p \neq 0$, the crystalline cohomology $H^*_{\mathrm{cris}}(X/W(K))$, a $W(K)$-module. Here $W(K)$ is the ring of p-typical Witt vectors over K.

There are some relations between these. For example, given an embedding $K \hookrightarrow \mathbb{C}$, there is an isomorphism

$$H^*(X(\mathbb{C}), \mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{C} \cong H^*_{\mathrm{dR}}(X) \otimes_K \mathbb{C}$$

given by integration of differential forms. A very concrete example is the following. Let $K = \mathbb{Q}$, $X = \mathbb{P}^1$. Then

$$H^2(X(\mathbb{C}), \mathbb{Z}) \cong \mathbb{Z}$$
$$H^2_{\mathrm{dR}}(X) \cong H^1(X, \Omega_X) \cong \mathbb{Q}$$

The group $H^1(X, \Omega_X)$ can be computed using Čech cohomology. We choose the covering $X = (X \setminus \{\infty\}) \cup (X \setminus \{0\})$. This gives us an exact sequence

$$k[z] \cdot dz \oplus k[z^{-1}] \cdot \frac{dz}{z^2} \to k[z, z^{-1}] \cdot dz \to H^1(X, \Omega_X) \to 0\,.$$

Then $H^1(X, \Omega_X)$ is generated by the image of $\frac{dz}{z}$. There is an isomorphism

$$H^2_{\mathrm{dR}}(X) \otimes_{\mathbb{Q}} \mathbb{C} \xrightarrow{\sim} H^2(X(\mathbb{C}), \mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{C}$$

which essentially integrates $\frac{dz}{z}$ along a loop γ around the origin. We have

$$\oint_\gamma \frac{dz}{z} = 2\pi i$$

so the above isomorphism is not defined over $\mathbb{Q}$. To get a natural isomorphism, we really needed to tensor with $\mathbb{C}$. The quantity $2\pi i$ is called a period.

(Note added after the lecture: applying the Mayer-Vietoris exact sequence to the covering mentioned above gives an isomorphism $H^1(X \setminus \{0, \infty\}) \xrightarrow{\sim} H^2(X)$ for both Betti and de Rham cohomology. This makes the concrete description of the comparison isomorphism more transparent.)

The above isomorphism is complex analytic in nature. One of the aims of the course is to explain a p-adic analogue of this result.

Definition 1.1.1. A *p-adic field* is a field K equipped with a discrete valuation, such that:

- K has characteristic zero.
- K is complete.
- The residue field of K has characteristic p, and is perfect.

This includes finite extensions of $\mathbb{Q}_p$, as well as the completion of the maximal unramified extension of $\mathbb{Q}_p$.

From now on, we will let K be a p-adic field.

The p-adic version of the de Rham comparison theorem is the following. It involves a "ring of periods" called B_{dR} that will be defined later.

Theorem 1.1.2. *Let X be a proper smooth algebraic variety over K. There is an isomorphism of filtered B_{dR}-vector spaces with* $\mathrm{Gal}(\overline{K}/K)$*-action*

$$H^*_{\mathrm{dR}}(X/K) \otimes_K B_{\mathrm{dR}} \cong H^*_{\text{ét}}(X_{\overline{K}}, \mathbb{Z}_p) \otimes_{\mathbb{Z}_p} B_{\mathrm{dR}} .$$

Remark 1.1.3. Let $\chi\colon G_K \to \mathbb{Z}_p^\times$ be the character defined by $g\zeta = \zeta^{\chi(g)}$ for all p-power roots of unity $\zeta \in \overline{K}^\times$. We call χ the *cyclotomic character*. The group $H^2_{\text{ét}}(\mathbb{P}^1_{\overline{K}}, \mathbb{Z}_p)$ is a free $\mathbb{Z}_p$-module of rank one, and G_K acts on this group by χ^{-1}. So the ring B_{dR} contains an element "$2\pi i$" and G_K acts on "$\mathbb{Q}_p \cdot (2\pi i)$" by the character χ^{-1}.

Let C be the completion of $\overline{K}$ with respect to the norm topology. We will show in a later lecture that G_K does not act by χ^{-1} on any nonzero subspace of C. So C does not contain any element "$2\pi i$", and Theorem 1.1.2 is not true if one replaces B_{dR} with C.

Remark 1.1.4. The field B_{dR} is actually the fraction field of a ring that is a completion of $\overline{K}$ for a topology that is finer than the norm topology.

Remark 1.1.5. It turns out that $B_{\mathrm{dR}}^{G_K} = K$, so we can recover the de Rham cohomology of X from its étale cohomology:

$$H^*_{\mathrm{dR}}(X/K) \cong \left(H^*_{\text{ét}}(X_{\overline{K}}, \mathbb{Z}_p) \otimes_{\mathbb{Z}_p} B_{\mathrm{dR}}\right)^{G_K} .$$

One can check that for any field K of characteristic zero,

$$\dim_K H^*_{\mathrm{dR}}(X/K) = \dim_{\mathbb{Q}_p} H^*_{\text{ét}}(X_{\overline{K}}, \mathbb{Q}_p) .$$

(Use the Lefschetz principle to reduce to the case where K embeds into $\mathbb{C}$ and then compare each side to the Betti cohomology.) An arbitrary finite-dimensional $\mathbb{Q}_p$-vector space representation V of G_K satisfies

$$\dim_K (V \otimes_{\mathbb{Q}_p} B_{\mathrm{dR}})^{G_K} \le \dim_{\mathbb{Q}_p} V .$$

If equality holds, we say that V is "B_{dR}-admissible" or "de Rham". In a later lecture, we will see examples where this inequality can be strict. Therefore, not all representations of G_K can appear in the étale cohomology of varieties over K.

Remark 1.1.6. Theorem 1.1.2 holds more generally if X is a proper smooth rigid analytic space over K.

1.2. **Some examples of Galois representations.** In p-adic Hodge theory, in addition to the cohomology of varieties, we also study representations of G_K, whether or not they come from the étale cohomology of a variety. Here are some examples of Galois representations.

- Fix an algebraic closure $\overline{K}$ of K. Let μ_{p^n} be the group of p^nth roots of unity in $\overline{K}$. Let
$$\mathbb{Z}_p(1) = \varprojlim_n \mu_{p^n} ;$$
it is a free $\mathbb{Z}_p$-module of rank 1.
- Local class field theory gives an explicit description of G_K^{ab}:
$$G_K^{\mathrm{ab}} \cong \hat{\mathbb{Z}} \times \mathcal{O}_K^\times \cong \hat{\mathbb{Z}} \times \mathbb{Z}_p^{[K:Q]} \times \mu(\mathcal{O}_K^\times) ,$$

where $\mu(\mathcal{O}_K^\times)$ is the group of roots of unity in $\mathcal{O}_K^\times$. So any representation of $\hat{\mathbb{Z}} \times \mathcal{O}_K^\times$ gives us a Galois representation. The representation $\mathbb{Z}_p(1)$ corresponds to the norm map

$$N_{K/\mathbb{Q}_p} \colon \mathcal{O}_K^\times \to \mathbb{Z}_p^\times .$$

- If X is a semiabelian variety over K (e.g. the multiplicative group $\mathbb{G}_m$ or an abelian variety), then we can consider its p-adic Tate module

$$T_p(X) := \varprojlim_n X(\overline{K})[p^n] .$$

  In particular,

$$\mathbb{Z}_p(1) = T_p(\mathbb{G}_m) .$$

- If X is an algebraic variety, then we can consider

$$H^i_{\text{ét}}(X_{\overline{K}}, \mathbb{Z}_p) .$$

  Note that if X is a semiabelian variety, then

$$H^i_{\text{ét}}(X_{\overline{K}}, \mathbb{Z}_p) \cong T_p(X)^* = \mathrm{Hom}_{\mathbb{Z}_p}(T_p(X), \mathbb{Z}_p) .$$

- If V and W are representations of G_K over some ring R, then $V \otimes_R W$ and $\mathrm{Hom}_R(V, W)$ are also representations of G_K.
- One can construct Galois representations by p-adic interpolation: if $\{V_n\}$ is an inverse system of $\mathbb{Z}/p^n\mathbb{Z}$-representations of G_K, then $V = \varprojlim_n V_n$ is a $\mathbb{Z}_p$-representation of G_K. There are examples where each V_n comes from the étale cohomology of some algebraic variety over K but V does not. This type of construction is used in the Langlands program.

1.3. **Outline of the course.** Here is an outline of some things that will be covered in the course:

- (φ, Γ)-modules. It is difficult to write down G_K explicitly, which makes it difficult to write down representations of G_K explicitly. We will introduce a category of (φ, Γ)-modules, which is equivalent to the category of G_K-modules, but whose objects are easier to write down.
- Perfectoid fields and the tilting correspondence. The tilting correspondence relates characteristic zero fields to characteristic p fields. For example, one can use the tilting correspondence to show that $\mathbb{Q}_p(\mu_{p^\infty})$ and $\mathbb{F}_p((t))$ have isomorphic Galois groups. Tilting allows one to use characteristic p methods to study p-adic fields. Tilting will also be used in the construction of "period rings" such as B_{dR}.
- The de Rham period ring B_{dR} and de Rham representations. The field B_{dR} contains all integrals of differentials on algebraic varieties over K. We say that a representation V of G_K is B_{dR}-admissible if $\dim_K (V \otimes_{\mathbb{Q}_p} B_{\mathrm{dR}})^{G_K} = \dim_{\mathbb{Q}_p} V$. In particular, representations appearing in the étale cohomology of algebraic varieties over K are de Rham.
- The crystalline period ring B_{cris} and crystalline representations. The crystalline period ring $B_{\mathrm{cris}} \subset B_{\mathrm{dR}}$ contains integrals of differentials on proper smooth algebraic varieties over $\mathcal{O}_K$. Representations appearing in proper smooth algebraic varieties with good reduction over K are B_{cris}-admissible.
- A sketch of a proof of the p-adic de Rham comparison theorem.

- (time permitting) The Fargues-Fontaine curve. This is a scheme of infinite type over $\mathbb{Q}_p$ that nonetheless behaves in many ways like a curve. Many constructions in p-adic Hodge theory have geometric interpretations involving this curve.
- (time permitting) Prismatic cohomology. This is a cohomology theory for formal schemes over $\mathcal{O}_K$ that specializes to étale, de Rham, and crystalline cohomology.

2. φ-modules and (φ, Γ)-modules

2.1. **Representations of characteristic p Galois groups.** It is difficult to write down Galois groups explicitly, which in turn makes it difficult to write down Galois representations explicitly. To deal with this problem, we will introduce φ-modules and (φ, Γ)-modules, which can be described more explicitly. We will show that categories of these modules are equivalent to categories of Galois representations.

There are many uses of (φ, Γ)-modules. Unfortunately, I will not be able to describe any in detail in this lecture. To give one example, they turn out to be useful for proving the p-adic local Langlands correspondence for $\mathrm{GL}_2(\mathbb{Q}_p)$.

We will exploit the fact that the absolute Galois groups of p-adic fields are closely related to the absolute Galois groups of characteristic p fields. For example, we have the following result, which will be proved in a later lecture.

Theorem 2.1.1. *The absolute Galois groups of $\mathbb{Q}_p(\mu_{p^\infty})$ and $\mathbb{F}_p((t))$ are isomorphic (as topological groups). Moreover, this isomorphism extends to an embedding $\mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p) \to \mathrm{Aut}(\mathbb{F}_p((t))^{\mathrm{sep}})$.*

Now let E be a field of characteristic p. Let $G_E = \mathrm{Gal}(E^{\mathrm{sep}}/E)$. Let $\varphi_E \colon E \to E$ be the Frobenius map $x \mapsto x^p$.

Given an E-module M, we write $\varphi_E^*(M)$ for its Frobenius pullback $E \otimes_{\varphi_E, E} M$. Any φ_E-semilinear map $\varphi_M \colon M \to M$ determines an E-linear map $\varphi_E^*(M) \to M$ by $e \otimes m \mapsto e\varphi_M(m)$.

Definition 2.1.2. A *φ-module* over E is a pair (M, φ_M), where M is a finite-dimensional E-vector space and φ_M is a φ_E-semilinear endomorphism. We say that (M, φ_M) is *étale* if the E-linear map $\varphi_E^*(M) \to M$ induced by φ is an isomorphism (equivalently, the image of φ_M generates M as an E-module).

We will denote the category of étale φ-modules over E by $\varphi\text{-}\mathrm{Mod}_E^{\text{ét}}$.

Let $\mathrm{Rep}_{\mathbb{F}_p}(G_E)$ denote the category of continuous finite-dimensional $\mathbb{F}_p$-vector space representations of G_E.

Theorem 2.1.3. *The functor $D_E \colon \mathrm{Rep}_{\mathbb{F}_p}(G_E) \to \varphi\text{-}\mathrm{Mod}_E^{\text{ét}}$ defined by*

$$V \mapsto (V \otimes_{\mathbb{F}_p} E^{\mathrm{sep}})^{G_E}.$$

and the functor $V_E \colon \varphi\text{-}\mathrm{Mod}_E^{\text{ét}} \to \mathrm{Rep}_{\mathbb{F}_p}(G_E)$ defined by

$$M \mapsto (M \otimes_E E^{\mathrm{sep}})^{\varphi=1}.$$

determine an equivalence of categories between $\mathrm{Rep}_{\mathbb{F}_p}(G_E)$ and $\varphi\text{-}\mathrm{Mod}_E^{\text{ét}}$.

Remark 2.1.4. One might think of Theorem 2.1.3 as a characteristic p version of the Riemann-Hilbert correspondence, with the Frobenius action replacing the connection. For a more geometric analogue, see [Kat73, Proposition 4.1.1].

Proof of Theorem 2.1.3. Let $V \in \mathrm{Rep}_{\mathbb{F}_p}(G_E)$. We will check that $D_E(V) \in \varphi\text{-}\mathrm{Mod}_E^{\text{ét}}$, and that there is a natural isomorphism $V_E(D_E(V)) \xrightarrow{\sim} V$. By Galois descent, the φ- and G_E-equivariant map

$$D_E(V) \otimes_E E^{\mathrm{sep}} \to V \otimes_{\mathbb{F}_p} E^{\mathrm{sep}} \tag{2.1.5}$$

is an isomorphism. Therefore, $\dim_E D_E(V) = \dim_{\mathbb{F}_p} V$; in particular, $D_E(V)$ is finite dimensional.

To show that $D_E(V)$ is étale, we just need to check that the matrix of Frobenius in some (equivalently, any) basis is invertible. By base change, the matrix of Frobenius on $D_E(V)$ is invertible iff the matrix of Frobenius on $D_E(V) \otimes_E E^{\mathrm{sep}} = V \otimes_{\mathbb{F}_p} E^{\mathrm{sep}}$ is invertible iff the matrix of Frobenius on V is invertible. The action of Frobenius on V is the identity.

Taking φ-invariants of (2.1.5) gives an isomorphism $V_E(D_E(V)) \xrightarrow{\sim} V$.

Now let $M \in \varphi\text{-}\mathrm{Mod}_E^{\text{ét}}$. We want to show that the natural map

$$E^{\mathrm{sep}} \otimes_{\mathbb{F}_p} V_E(M) \to E^{\mathrm{sep}} \otimes_E M \tag{2.1.6}$$

is an isomorphism. First we will show that it is injective. It suffices to show that if some vectors in $V_E(M) = (E^{\mathrm{sep}} \otimes_E M)^{\varphi=1}$ are linearly independent over $\mathbb{F}_p$, then they are also linearly independent over E^{sep}. Suppose that there is a minimal counterexample $v_1, \dots, v_r \in V_E(M)$, with $\sum_{i=1}^r a_i v_i = 0$ for $a_i \in E^{\mathrm{sep}}$. WLOG we may take $a_1 = 1$. Using $\varphi(v_i) = v_i$ and $\varphi(a_1) = a_1$, we obtain $0 = \sum_{i=2}^r (a_i - \varphi(a_i))v_i$. By minimality of the counterexample, we must have $(a_i - \varphi(a_i)) = 0$ for all i. Hence $a_i \in \mathbb{F}_p$ for all i, which is a contradiction. Note that we did not need to use the fact that M is étale to prove injectivity.

Now we show that (2.1.6) is surjective. Let c_{ij} be the matrix coefficients of Frobenius in some basis. Let X be the scheme over E defined by the equations

$$x_i^p = \sum_j c_{ij} x_j \, .$$

Surjectivity of (2.1.6) is equivalent to $|X(E_s)| = p^{\dim_E M}$. Since the degree of X over E is $p^{\dim_E M}$, it suffices to show that X is étale over E, or equivalently that $\Omega_{X/E} = 0$. The module $\Omega_{X/E}$ is generated by the dx_i subject to the relations $\sum_j c_{ij} x_j = 0$. Since the c_{ij} define an invertible matrix, $\Omega_{X/E} = 0$. This concludes the proof that (2.1.6) is an isomorphism.

From (2.1.6), we see that $V_E(M)$ is finite dimensional over $\mathbb{F}_p$. Then $V_E(M) = (M \otimes_E F)^{\varphi=1}$ for some finite separable extension F/E, so the G_E-action on $V_E(M)$ is continuous. Therefore $V_E(M) \in \mathrm{Rep}_{\mathbb{F}_p} G_E$. Taking Galois invariants of (2.1.6) gives an isomorphism $D_E(V_E(M)) \xrightarrow{\sim} M$. Hence we have shown that the functors D_E and V_E are essential inverses of each other. □

Now we turn our attention to $\mathbb{Z}_p$-representations of G_E. Let $\mathrm{Rep}_{\mathbb{Z}_p} G_E$ denote the category of finitely generated (not necessarily free) $\mathbb{Z}_p$-modules with continuous G_E-action.

One can show that for any E, there is a complete discrete valuation ring $\mathcal{O}_{\mathcal{E}}$ such that the residue field of $\mathcal{O}_{\mathcal{E}}$ is E, and p is a uniformizer of $\mathcal{O}_{\mathcal{E}}$. Such a ring is called a Cohen ring for E. It is unique up to isomorphism. We can also find a lift of Frobenius to $\mathcal{O}_{\mathcal{E}}$.

Example 2.1.7. If $E = \mathbb{F}_{p^n}$, then $\mathcal{O}_\mathcal{E}$ is the ring of integers of $\mathbb{Q}_{p^n}$, the unramified extension of $\mathbb{Q}_p$ of degree n. More generally, if E is perfect, then $\mathcal{O}_\mathcal{E} \cong W(E)$, the ring of p-typical Witt vectors over E. If $E = \mathbb{F}_p((T))$, then we can take

$$\mathcal{O}_\mathcal{E} = \left\{ \sum_{n=-\infty}^{\infty} a_n T^n \middle| a_n \in \mathbb{Z}_p, \lim_{n\to-\infty} a_n = 0 \right\}.$$

A commonly used choice of Frobenius action is $T \mapsto (1+T)^p - 1$.

Definition 2.1.8. The category $\varphi\text{-Mod}^{\text{ét}}_{\mathcal{O}_\mathcal{E}}$ of étale φ-modules over $\mathcal{O}_\mathcal{E}$ consists of pairs (M, φ_M) where M is a finitely generated $\mathcal{O}_\mathcal{E}$-module and φ_M is a φ-semilinear endomorphism of M such that $\varphi^*_{\mathcal{O}_\mathcal{E}}(M) \to M$ is an isomorphism.

Now let $\check{\mathcal{O}}_\mathcal{E} = \widehat{\mathcal{O}^{\text{sh}}_\mathcal{E}}$ be the completion of the strict henselization of $\mathcal{O}_\mathcal{E}$. There is a unique continuous Frobenius on $\check{\mathcal{O}}_\mathcal{E}$ that extends the Frobenius on $\mathcal{O}_\mathcal{E}$ and E^{sep}.

Theorem 2.1.9. *The functor* $D_\mathcal{E}\colon \operatorname{Rep}_{\mathbb{Z}_p} G_E \to \varphi\text{-Mod}^{\text{ét}}_{\mathcal{O}_\mathcal{E}}$ *defined by*

$$V \mapsto (V \otimes_{\mathbb{Z}_p} \check{\mathcal{O}}_\mathcal{E})^{G_E}$$

and the functor $V_\mathcal{E}\colon \varphi\text{-Mod}^{\text{ét}}_{\mathcal{O}_\mathcal{E}} \to \operatorname{Rep}_{\mathbb{Z}_p} G_E$ *defined by*

$$M \mapsto (M \otimes_{\mathcal{O}_\mathcal{E}} \check{\mathcal{O}}_\mathcal{E})^{\varphi=1}$$

determine an equivalence of categories between $\operatorname{Rep}_{\mathbb{Z}_p} G_E$ *and* $\varphi\text{-Mod}^{\text{ét}}_{\mathcal{O}_\mathcal{E}}$.

Finally, we consider $\mathbb{Q}_p$-representations of G_E. Let $\operatorname{Rep}_{\mathbb{Q}_p} G_E$ denote the category of finite-dimensional $\mathbb{Q}_p$-vector space representation of G_E. Let $\mathcal{E} := \mathcal{O}_\mathcal{E}[1/p], \check{\mathcal{E}} := \check{\mathcal{O}}_\mathcal{E}[1/p]$.

Definition 2.1.10. The category $\varphi\text{-Mod}^{\text{ét}}_{\mathcal{E}}$ of étale φ-modules over $\mathcal{E}$ consists of pairs (M, φ_M) where M is a finite-dimensional $\mathcal{E}$-vector space and φ_M is a φ-semilinear endomorphism of M such that $\varphi^*_\mathcal{E}(M) \to M$ is an isomorphism, and M admits a φ_M-stable $\mathcal{O}_\mathcal{E}$-lattice.

Theorem 2.1.11. *The functor* $D_\mathcal{E}\colon \operatorname{Rep}_{\mathbb{Z}_p} G_E \to \varphi\text{-Mod}^{\text{ét}}_{\mathcal{E}}$ *defined by*

$$V \mapsto (V \otimes_{\mathbb{Q}_p} \check{\mathcal{E}})^{G_E}$$

and the functor $V_\mathcal{E}\colon \varphi\text{-Mod}^{\text{ét}}_{\mathcal{E}} \to \operatorname{Rep}_{\mathbb{Z}_p} G_E$ *defined by*

$$M \mapsto (M \otimes_{\mathcal{E}} \check{\mathcal{E}})^{\varphi=1}$$

determine an equivalence of categories between $\operatorname{Rep}_{\mathbb{Q}_p} G_E$ *and* $\varphi\text{-Mod}^{\text{ét}}_{\mathcal{E}}$.

3. (φ,Γ)-modules, Perfectoid fields, tilting, Witt vectors

3.1. **(φ,Γ)-modules.** The Galois group $\operatorname{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$ contains the closed normal subgroup $\operatorname{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p(\mu_{p^\infty})) \cong \operatorname{Gal}(\mathbb{F}_p((t))^{\text{sep}}/\mathbb{F}_p((t)))$. Moreover, this isomorphism can be extended to a map

$$\operatorname{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p) \hookrightarrow \operatorname{Aut}(\mathbb{F}_p((t))^{\text{sep}}).$$

This motivates us to consider the following setup.

Let G be a profinite group containing G_E as a closed normal subgroup. Let $\Gamma = G/G_E$. Suppose that we are given a continuous action of Γ on $\mathcal{O}_\mathcal{E}$. There is an induced action of G on $\check{\mathcal{O}}_\mathcal{E}$ (again because compatible endomorphisms on E^{sep} and $\mathcal{O}_\mathcal{E}$ extend uniquely to $\check{\mathcal{O}}_\mathcal{E}$).

Definition 3.1.1. A (φ, Γ)-*module* over $\mathcal{O}_{\mathcal{E}}$ is a φ-module over $\mathcal{O}_{\mathcal{E}}$ equipped with a semilinear Γ-action commuting with the φ-action. We say that a (φ, Γ)-module is *étale* if it is étale as a φ-module.

Write $(\varphi, \Gamma)\text{-}\mathrm{Mod}^{\text{ét}}_{\mathcal{O}_{\mathcal{E}}}$ for the category of étale (φ, Γ)-modules over $\mathcal{O}_{\mathcal{E}}$, and write $\mathrm{Rep}_{\mathbb{Z}_p} G$ for the category of finitely generated $\mathbb{Z}_p$-modules with G-action.

Theorem 3.1.2. *The functor* $D_{\mathcal{E}} \colon \mathrm{Rep}_{\mathbb{Z}_p} G \to (\varphi, \Gamma)\text{-}\mathrm{Mod}^{\text{ét}}_{\mathcal{O}_{\mathcal{E}}}$ *defined by*

$$V \mapsto (V \otimes_{\mathbb{Z}_p} \check{\mathcal{O}}_{\mathcal{E}})^{G_E}$$

and the functor $V_{\mathcal{E}} \colon (\varphi, \Gamma)\text{-}\mathrm{Mod}^{\text{ét}}_{\mathcal{O}_{\mathcal{E}}} \to \mathrm{Rep}_{\mathbb{Z}_p} G$ *defined by*

$$M \mapsto (M \otimes_{\mathcal{O}_{\mathcal{E}}} \check{\mathcal{O}}_{\mathcal{E}})^{\varphi=1}$$

determine an equivalence of categories between $\mathrm{Rep}_{\mathbb{Z}_p} G$ *and* $(\varphi, \Gamma)\text{-}\mathrm{Mod}^{\text{ét}}_{\mathcal{O}_{\mathcal{E}}}$.

A similar result holds for $\mathbb{F}_p$- and $\mathbb{Q}_p$-representations.

Example 3.1.3. Let $G = G_{\mathbb{Q}_p}$, $E = \mathbb{F}_p((T))$, and embed G in $\mathrm{Aut}\, E^{\mathrm{sep}}$ using Theorem 2.1.1. We have $\Gamma = \mathrm{Gal}(\mathbb{Q}_p(\mu_{p^\infty})/\mathbb{Q}_p) \cong \mathbb{Z}_p^\times$. The action of Γ on $\mathbb{F}_p((T))$ is given by

$$\gamma \cdot T = (1+T)^\gamma - 1\,.$$

The action of Γ on $\mathcal{O}_{\mathcal{E}}$ can then also be taken to be $\gamma \cdot T = (1+T)^\gamma - 1$.

3.2. **Perfectoid fields.** We have claimed that $\mathbb{Q}_p(\mu_{p^\infty})$ and $\mathbb{F}_p((t))$ have isomorphic Galois groups. To prove the isomorphism, we will make use of the concept of perfectoid fields and the tilting correspondence.

Definition 3.2.1. A *nonarchimedean field* K is a field that is complete with respect to a nontrivial nonarchimedean metric $|\cdot|$. We will write

$$\mathcal{O}_K := \{x \in K | |x| \le 1\}$$

$$\mathfrak{m}_K := \{x \in K | |x| < 1\}$$

Definition 3.2.2. A nonarchimedean field K of residue characteristic p is *perfectoid* if its value group is nondiscrete and the Frobenius map

$$\Phi \colon \mathcal{O}_K/p \to \mathcal{O}_K/p$$

is surjective.

Remark 3.2.3. Like most references but unlike [Ked15], we do not require that K have characteristic zero.

Example 3.2.4.

- The field $\mathbb{C}_p$ is perfectoid. More generally, any algebraically closed nonarchimedean field of residue characteristic p is perfectoid.
- A nonarchimedean field of characteristic p is perfectoid if and only if it is perfect.

Lemma 3.2.5. *The field* $\mathbb{Q}_p^{\mathrm{cyc}} := \widehat{\mathbb{Q}_p(\mu_{p^\infty})}$ *is perfectoid.*

Proof. Let $\{\zeta_{p^n}\}_{n\ge 0}$ denote a system of p-power roots of unity. Note that

$$\mathcal{O}_{\mathbb{Q}_p^{\mathrm{cyc}}}/p \cong \varinjlim_n \mathbb{Z}_p[\zeta_{p^n}]/p\,.$$

Recall that the minimal polynomial of $\zeta_{p^n} - 1$ is

$$\frac{(1+x)^{p^n} - 1}{(1+x)^{p^{n-1}} - 1} \equiv x^{p^{n-1}(p-1)} \pmod{p}.$$

So we can write $\mathbb{Z}_p[\zeta_{p^n}]/p \cong \mathbb{F}_p[x]/x^{p^{n-1}(p-1)} \cong \mathbb{F}_p[t^{p^{-n}}]/t^{(p-1)/p}$, and there is a commutative diagram

$$\begin{array}{ccc} \mathbb{F}_p[t^{p^{-n}}]/t^{(p-1)/p} & \hookrightarrow & \mathbb{F}_p[t^{p^{-n-1}}]/t^{(p-1)/p} \\ \downarrow \sim & & \downarrow \sim \\ \mathbb{Z}_p[\zeta_{p^n}]/p & \hookrightarrow & \mathbb{Z}_p[\zeta_{p^{n+1}}]/p\,. \end{array}$$

The Frobenius on $\varinjlim_n \mathbb{F}_p[t^{p^{-n}}]/t^{(p-1)/p}$ is clearly surjective. □

Definition 3.2.6. Let K be a perfectoid field. The *tilt* of K, denoted $K^\flat$, is defined by

$$K^\flat := \varprojlim_{z \mapsto z^p} K\,.$$

Define addition on $K^\flat$ by $(a_n) + (b_n) = (c_n)$, where

$$c_n = \lim_{m \to \infty} (a_{m+n} + b_{m+n})^{p^m}$$

and define multiplication on $K^\flat$ by componentwise multiplication.

Define a homomorphism of multiplicative monoids $\sharp\colon K^\flat \to K$ by $(a_n)^\sharp = a_0$.

Lemma 3.2.7.

(1) *The limit in Definition 3.2.6 exists.*
(2) *$K^\flat$ is a field of characteristic p.*
(3) *The function $(a_n) \mapsto |(a_n)^\sharp| = |a_0|$ is a nonarchimedean norm on $K^\flat$, and $K^\flat$ is a perfectoid field.*
(4) *We have*

$$\mathcal{O}_{K^\flat} = \varprojlim_{z \mapsto z^p} \mathcal{O}_K \cong \varprojlim_{\Phi} \mathcal{O}_K/p\,.$$

(5) $|K^\times| = |K^{\flat\times}|$.

Proof. Left as an exercise to the reader. Parts (1) and (4) use the following lemma. □

Lemma 3.2.8. *Let R be a ring, let $x, y \in R$, and let n be a positive integer. If $x \equiv y \pmod{p^n}$, then $x^p \equiv y^p \pmod{p^{n+1}}$.*

Example 3.2.9. From the analysis of Lemmas 3.2.5 and 3.2.7, we see that $(\mathbb{Q}_p^{\mathrm{cyc}})^\flat$ is isomorphic to $\mathbb{F}_p((t^{p^{-\infty}}))$, the completion of $\varinjlim_n \mathbb{F}_p((t^{p^{-n}}))$.

Definition 3.2.10. Let K be a perfectoid field of characteristic p. An *untilt* of K is a perfectoid field $K^\sharp$, along with an isomorphism $(K^\sharp)^\flat \xrightarrow{\sim} K$.

We would like to classify the untilts of a given characteristic p perfectoid field K. In order to do that, we will need to introduce the ring $W(\mathcal{O}_K)$.

3.3. **Witt vectors.**

Definition 3.3.1. A *strict p-ring* is a ring R such that R is p-adically complete and separated, R/pR is a perfect $\mathbb{F}_p$-algebra, and p is not a zero divisor in R.

Example 3.3.2. If K is the completion of an unramified extension of $\mathbb{Q}_p$, then $\mathcal{O}_K$ is a strict p-ring. The p-adic completion of $\mathbb{Z}[x^{p^{-\infty}}]$ is also a strict p-ring.

The main goal of this section is to prove the following theorem.

Theorem 3.3.3. *The functor $A \mapsto A/pA$ from strict p-rings to perfect $\mathbb{F}_p$-algebras is an equivalence of categories.*

We will write W for the functor from perfect $\mathbb{F}_p$-algebras to strict p-rings determined by the above equivalence. For R a perfect $\mathbb{F}_p$-algebra, the ring $W(R)$ is called the ring of p-typical Witt vectors of R.

Lemma 3.3.4. *Let R be a strict p-ring.*

(1) *There is a unique section $[\cdot]$ of the reduction map $R \to R/pR$ that is a homomorphism of multiplicative monoids.*
(2) *Every element of R can be written uniquely in the form*

$$\sum_{n=0}^{\infty} p^n [a_n], \quad a_n \in R/pR\,.$$

Proof. Left as an exercise to the reader. The first part uses Lemma 3.2.8. □

Lemma 3.3.5. *Let R be a strict p-ring, and let $a, b \in R$. Suppose that $a = \sum_{n=0}^{\infty}[a_n]p^n$, $b = \sum_{n=0}^{\infty}[b_n]p^n$, $a+b = \sum_{n=0}^{\infty}[s_n]p^n$, $ab = \sum_{n=0}^{\infty}[t_n]p^n$. Then s_n and t_n are polynomials in the $a_i^{p^{i-n}}$, $b_i^{p^{i-n}}$ for $0 \le i \le n$. Furthermore, s_n is homogeneous of degree 1 (where each a_i and b_i has degree 1), and t_n is homogeneous in the a_i and b_i separately, each of degree 1.*

Proof. Repeatedly use the identity

$$[x+y] \equiv ([x^{p^{-n}}] + [y^{p^{-n}}])^{p^n} \pmod{p^{n+1}}\,,$$

which follows from Lemma 3.2.8. □

Proposition 3.3.6. *Let R be a strict p-ring, and let S be a p-adically complete ring. Let $\sharp\colon R/pR \to S$ be a multiplicative map that induces a homomorphism of rings $R/pR \to S/pS$. Then the formula*

$$\Theta\left(\sum_{n=0}^{\infty} p^n [x_n]\right) = \sum_{n=0}^{\infty} p^n x_n^{\sharp}$$

defines a p-adically continuous homomorphism $\Theta\colon R \to S$ such that $\Theta \circ [\cdot] = \sharp$.

We are especially interested in applying this result in the case where $R = W(\mathcal{O}_{K^\flat})$ and $S = \mathcal{O}_K$ for some perfectoid field K.

Proof. See [Ked15, Lemma 1.1.6]. □

Proof of Theorem 3.3.3. Full faithfulness follows from Proposition 3.3.6.

To prove essential surjectivity, let $\overline{R}$ be a perfect ring of characteristic p, and write $\overline{R} = \mathbb{F}_p[X^{-p^{\infty}}]/\overline{I}$ for some set X and ideal $\overline{I} \subset \mathbb{F}_p[X^{-p^{\infty}}]$. Let R_0 be the p-adic completion of $\mathbb{Z}_p[X^{-p^{\infty}}]$; then one can check that R_0 is a strict p-ring and

$R_0/pR_0 = \mathbb{F}_p[X^{-p^\infty}]$. Let $I \subset R_0$ be the set of elements of the form $\sum_{n=0}^{\infty} p^n[x_n]$ with $x_n \in \overline{I}$. Then one can check that I is an ideal of R_0 and $R := R_0/I$ is a strict p-ring with $\overline{R} = R/pR$. □

4. Untilting

4.1. Untilts and $W(\mathcal{O}_{K^\flat})$.

Let K be a perfectoid field.

Definition 4.1.1. An ideal I of $W(\mathcal{O}_{K^\flat})$ is *primitive of degree* 1 if it is generated by an element of the form $p + [\pi]\alpha$ for some $\pi \in \mathfrak{m}_{K^\flat}$, $\alpha \in W(\mathcal{O}_{K^\flat})$.

Proposition 4.1.2. *The map*

$$\Theta\colon W(\mathcal{O}_{K^\flat}) \to \mathcal{O}_K$$

defined in Proposition 3.3.6 has the following properties:

(1) Θ *is surjective.*
(2) $\ker\Theta$ *is primitive of degree* 1.

Proof. By Lemma 3.2.7(4), the map $\sharp$ is surjective mod p. So by successive approximation, every element of $\mathcal{O}_K$ can be written as $\sum_{n=0}^{\infty} a_n^\sharp p^n$ for some $a_n \in \mathcal{O}_{K^\flat}$. Therefore, Θ is surjective.

If K has characteristic p, then $\ker\Theta = (p)$ is primitive of degree 1. Now suppose K has characteristic 0. Choose $\pi^\flat \in \mathcal{O}_{K^\flat}$ so that $\pi := (\pi^\flat)^\sharp$ satisfies $|\pi| = |p|$. Choose $x \in W(\mathcal{O}_{K^\flat})$ satisfying $\Theta(x) = -p/\pi$. Since $\Theta(x)$ is a unit of K, the constant term in the Teichmuller expansion of x must be a unit; then x is also a unit. Let $\xi = p + [\pi^\flat]x$; then $\xi \in \ker\Theta$. We claim that in fact ξ generates $\ker\Theta$. Observe that $\ker\Theta \subseteq ([\pi^\flat], p) = (\xi, p)$. So any element of $\ker\Theta$ can be written as $a\xi + bp$ with $\Theta(pb) = p\Theta(b) = 0$. Since p is not a zero divisor in $\mathcal{O}_K$, we get $\Theta(b) = 0$. By successive p-adic approximation, we see that $\ker\Theta = (\xi)$. □

Remark 4.1.3. If you find it dissatisfying that we used a separate argument for $p = 0$, see [BMS18, Lemma 3.2ii, Lemma 3.10] for a version of the argument that generalizes better. Essentially, the idea is to use Lemma 3.3.5 to prove that $W(\mathcal{O}_{K^\flat})$ is complete for the $[\pi^\flat]$-adic topology; then we can use $[\pi^\flat]$-adic approximation and we can assume $|p| \le |\pi| < 1$ instead of $|\pi| = |p|$.

Proposition 4.1.4. *The category of perfectoid fields is equivalent to the category of pairs (K, I), where K is a perfectoid field of characteristic p and $I \subset W(\mathcal{O}_K)$ is an ideal that is primitive of degree* 1.

Proof. See [Ked15, Theorem 1.4.13]. □

Corollary 4.1.5. *Let K be a perfectoid field. Then tilting induces an equivalence of categories between perfectoid extensions of K and perfectoid extensions of $K^\flat$.*

Moreover, if L/K is an extension of perfectoid fields, then L/K is finite iff $L^\flat/K^\flat$ is finite.

Lemma 4.1.6. *If K is a perfectoid field and $K^\flat$ is algebraically closed, then so is K.*

Proof. Let $P(X) = X^d + a_{d-1}X^{d-1} + \cdots + a_0 \in K[X]$ be a monic irreducible polynomial. Since $K^\flat$ is algebraically closed, $|K^{\flat\times}|$ is a $\mathbb{Q}$-vector space, so $|K^\times|$ is as well. Therefore, by scaling the variable, we may assume that $a_0 \in \mathcal{O}_K^\times$. Since P is irreducible, its Newton polygon must be a straight line, so $a_i \in \mathcal{O}_K$ for all i.

Let $Q(X) \in \mathcal{O}_K^\flat[X]$ be a monic polynomial such that P and Q have the same image in $(\mathcal{O}_K/p\mathcal{O}_K)[X]$. Let $y \in \mathcal{O}_{K^\flat}$ be a root of $Q(X)$. Then $p \mid P(y^\sharp)$. If $P(y^\sharp) \neq 0$, choose $c \in \mathcal{O}_K$ so that $|c|^d = |P(y^\sharp)|$. Then replace $P(X)$ with $c^{-d}P(cX + y^\sharp)$. By repeating this process, we find a sequence of elements of $\mathcal{O}_K$ converging to a root of P. □

Proposition 4.1.7. *Any finite extension of a perfectoid field is perfectoid.*

Proof. Let K be a perfectoid field, and let $C^\flat$ be the completion of an algebraic closure of $K^\flat$. By Corollary 4.1.5, $C^\flat$ has an untilt C over K. Furthermore, C is algebraically closed by Lemma 4.1.6. Let C_0 be the union of the untilts of all finite extensions of $K^\flat$; then C_0 is dense in C since the union of all finite extensions of $C^\flat$ is dense in $C^\flat$. It follows from Krasner's lemma that a dense subfield of an algebraically closed nonarchimedean field is separably closed. Then C_0 must contain all finite extensions of K. So any finite extension L/K is contained in a Galois extension M/K that is an untilt of some $M^\flat/K^\flat$. By Galois theory, any subfield of M containing K must be the untilt of a subfield of $M^\flat$ containing $K^\flat$. □

Theorem 4.1.8. *Let K be a perfectoid field. There is an equivalence of categories between finite extensions of K and finite extensions of $K^\flat$.*

Hence there is an injection

$$\mathrm{Aut}_{\mathrm{cts}}(\overline{K}) \hookrightarrow \mathrm{Aut}_{\mathrm{cts}}(\overline{K^\flat})$$

inducing an isomorphism

$$\mathrm{Gal}(\overline{K}/K) \cong \mathrm{Gal}(\overline{K^\flat}/K^\flat)\,.$$

Proof. Combine Corollary 4.1.5 and Proposition 4.1.7. □

Corollary 4.1.9. *The fields $\mathbb{Q}_p(\mu_{p^\infty})$, $\mathbb{Q}_p^{\mathrm{cyc}} = \widehat{\mathbb{Q}_p(\mu_{p^\infty})}$, $\mathbb{F}_p((t^{p^{-\infty}}))$, and $\mathbb{F}_p((t))$ have isomorphic Galois groups.*

Proof. By the above theorem, $\mathbb{Q}_p^{\mathrm{cyc}} = \widehat{\mathbb{Q}_p(\mu_{p^\infty})}$ and $\mathbb{F}_p((t^{p^{-\infty}}))$ have isomorphic Galois groups. By Krasner's lemma, taking completions does not change the Galois group, and taking perfections also does not change the Galois group. □

4.2. **Ax–Sen–Tate theorem.** Let K be a p-adic field, and let $C := \widehat{\overline{K}}$ be the completion of its algebraic closure with respect to the norm topology. In a previous lecture, I claimed that there is no element "$2\pi i$" in C so that G_K acts on "$2\pi i$" $\cdot\,\mathbb{Q}_p$ by the cyclotomic character χ.

Theorem 4.2.1. *$C^{G_K} = K$.*

Let $K_\infty := K(\mu_{p^\infty})$, $K^{\mathrm{cyc}} := \widehat{K_\infty}$, $\Gamma := \mathrm{Gal}(K_\infty/K)$. If $\chi\colon \Gamma \to K^\times$ has infinite order, then $C(\chi)^{G_K} = 0$.

Remark 4.2.2. One can also show that $H^1(G_K, C)$ is a one-dimensional K-vector space and that $H^1(G_K, C(\chi)) = 0$. In the interest of space, we omit the proof. See [Tat67, §3].

For an elementary (but calculation-heavy) proof that $C^{G_K} = K$, see [Ax70] or [FO, Proposition 3.8].

The proof breaks down into the following steps.

Lemma 4.2.3. *The field K^{cyc} is perfectoid.*

Proof. Let k be the residue field of K. Then $W(k)[1/p]^{\mathrm{cyc}}$ is perfectoid by the same argument as in Lemma 3.2.5. Since K is finite extension of $W(k)[1/p]$, the result follows from Lemma 4.1.7. □

Proposition 4.2.4. *If L is a perfectoid field, then $(\widehat{\overline{L}})^{G_L} = L$. In particular, $C^{\mathrm{Gal}(\overline{K}/K_\infty)} = K^{\mathrm{cyc}}$.*

Proposition 4.2.5. $(K^{\mathrm{cyc}})^\Gamma = K$.
If $\chi\colon \Gamma \to K^\times$ has infinite order, then $K^{\mathrm{cyc}}(\chi)^\Gamma = 0$.

To prove Proposition 4.2.4, we will need a few lemmas.

Lemma 4.2.6. *Let M/L be a finite extension of perfectoid fields. Then $\mathrm{tr}_{M/L}(\mathfrak{m}_M) = \mathfrak{m}_L$.*

Proof. Since $M^\flat/L^\flat$ is separable, $\mathrm{tr}_{M^\flat/L^\flat}(\mathfrak{m}_{M^\flat})$ is a nonzero ideal of $\mathcal{O}_L$. By applying the inverse of Frobenius, we see that it must be all of $\mathfrak{m}_{L^\flat}$. Since there are compatible surjective ring homomorphisms $\mathcal{O}_{M^\flat} \twoheadrightarrow \mathcal{O}_M/p\mathcal{O}_M$, $\mathcal{O}_{L^\flat} \twoheadrightarrow \mathcal{O}_L/p\mathcal{O}_L$, this implies that $\mathrm{tr}_{M/L}(\mathfrak{m}_M) = \mathfrak{m}_L$. □

Lemma 4.2.7. *Let L be a perfectoid field, and let $y \in \overline{L}$. Let $c > 1$ be a real number. Then there exists $z \in L$ so that*

$$|y - z| \le c \max_{\sigma \in G_L} |\sigma y - y| .$$

Proof. Choose a finite extension M of L containing y. We will write tr for the trace from M to M. By Lemma 4.2.6, we can find $x \in M$ with $|x| < 1$, $|\operatorname{tr} x| \ge c^{-1}$. Let $z = \frac{\mathrm{tr}(xy)}{\mathrm{tr}\, y}$. Then

$$y - z = \frac{\sum_{\sigma \in \mathrm{Gal}(M/L)} (\sigma x)(y - \sigma y)}{\operatorname{tr} x} .$$

Hence $|y - z| \le c \max_{\sigma \in H_K} |\sigma y - y|$, as desired. □

Proof of Proposition 4.2.4. Let $x \in (\widehat{\overline{L}})^{G_L}$. Then for any real $\epsilon > 0$, we can find $y \in \overline{L}$ so that $|x - y| < \epsilon$ and $|\sigma y - y| < \epsilon$ for all $\sigma \in G_L$. By Lemma 4.2.7, we can find $z \in L$ so that $|y - z| \le c\epsilon$; hence $|x - z| < c\epsilon$. Since this is true for any ϵ, $x \in L$. □

5. Ax–Sen–Tate continued, B_{dR}

5.1. **Ax–Sen–Tate continued.**

Proof of Proposition 4.2.5. There is a "normalized trace" map $t\colon K_\infty \to K$ satisfying $t|_L = \frac{1}{[L:K]} \mathrm{tr}_{L/K}$ for every finite extension L/K inside K_∞. We claim that t is continuous. Indeed, this can be checked explicitly if $K = W(k)[1/p]$, and in general we can use the fact that K_∞ is a direct summand of $W(k)[1/p]_\infty \otimes_{W(k)[1/p]} K$. Therefore, we can extend t to a continuous map $K^{\mathrm{cyc}} \to K$, which we will also denote by t. Since t is idempotent, we get a direct sum decomposition $K^{\mathrm{cyc}} = K \oplus \ker t$.

We claim that for $x \in K_\infty$,

$$|x - t(x)| \le |p|^{-1} |x - \gamma x| ,$$

and hence $1 - \gamma$ has a continuous inverse on $\ker t$. There is no harm in replacing K by a finite cyclotomic extension, so we may assume $\Gamma \cong \mathbb{Z}_p$. Choose a generator

$\gamma \in \Gamma$. For each n, let K_n be the fixed field of $p^n\Gamma$. We will prove the inequality on each K_n by induction. The base case $n = 0$ is trivial. Since $1-\gamma$ divides $p - \left(1 + \gamma^{p^{n-1}} + \cdots + \gamma^{p^{n-1}(p-1)}\right)$, we have

$$\left|x - p^{-1}\operatorname{tr}_{K_n/K_{n-1}} x\right| \le |p^{-1}|\,|((1-\gamma)x|\ .$$

By the induction hypothesis,

$$\left|p^{-1}\operatorname{tr}_{K_n/K_{n-1}} - t(x)\right| \le |p^{-1}|\left|((1-\gamma)(p^{-1}\operatorname{tr}_{K_n/K_{n-1}} x)\right| \le |p^{-1}||(1-\gamma)x|\,,$$

where we used the fact that $1-\gamma$ commutes with the normalized trace in the last inequality. Then the claim follows from the triangle inequality.

So we have shown that $(\ker t)^\Gamma = 0$, and $(K^{\mathrm{cyc}})^\Gamma = K$.

Finally, suppose that $\chi\colon \Gamma \to K^\times$ is a character of infinite order. We will show that $K^{\mathrm{cyc}}(\chi^{-1})^\Gamma = 0$. Since Γ is p-adically complete, we must have $|\chi(\gamma) - 1| < 1$. After replacing K by a finite cyclotomic extension (and thus replacing γ by a power), we may assume that $|\chi(\gamma) - 1| < |p|$. On $\ker t$,

$$\gamma - \chi(\gamma) = (\gamma - 1)(1 - (\chi(\gamma) - 1)(\gamma - 1)^{-1})$$

and $(1 - (\chi(\gamma) - 1)(\gamma - 1)^{-1})^{-1}$ has a convergent power series, so $\gamma - \chi(\gamma)$ is invertible. On K, $\gamma - \chi(\gamma) = 1 - \chi(\gamma)$ is invertible since χ has infinite order. □

5.2. **The ring B_{dR}.** Now we define the ring B_{dR} that appeared in Theorem 1.1.2.

Let K be a p-adic field, and let C be the completion of its algebraic closure with respect to the norm topology. We define the ring

$$A_{\mathrm{inf}} := W(\mathcal{O}_{C^\flat})\,.$$

By Proposition 3.3.6, there is a homomorphism

$$\Theta\colon A_{\mathrm{inf}} \to \mathcal{O}_C\,.$$

We will consider the localization

$$\Theta_{\mathbb{Q}}\colon A_{\mathrm{inf}}[1/p] \to C\,.$$

Lemma 5.2.1. *For each positive integer n, $(\ker\Theta_{\mathbb{Q}})^n \cap A_{\mathrm{inf}} = (\ker\Theta)^n$, and $\bigcap_n (\ker\Theta_{\mathbb{Q}})^n = 0$.*

Let

$$B_{\mathrm{dR}}^+ := \varprojlim_n A_{\mathrm{inf}}[1/p]/(\ker\Theta_{\mathbb{Q}})^n\,.$$

Then B_{dR}^+ is a complete discrete valuation ring with residue field C.

Proof. Left as an exercise to the reader. □

Define

$$B_{\mathrm{dR}} := \operatorname{Frac} B_{\mathrm{dR}}^+\,.$$

Define a decreasing filtration on B_{dR} by letting $\operatorname{Fil}^i B_{\mathrm{dR}}$ be the fractional ideal $(\ker\Theta_{\mathbb{Q}})^i$. Now we will define an element $t \in B_{\mathrm{dR}}^+$ that is the p-adic analogue of $2\pi i$. Let $\epsilon \in \mathcal{O}_{C^\flat}$ be an element with $\epsilon_0 = 1$, $\epsilon_1 \ne 1$. Then $[\epsilon] - 1 \in \ker\Theta$, so it makes sense to define

$$t := \log[\epsilon] = \sum_{n=1}^{\infty} (-1)^{n+1}\frac{([\epsilon]-1)^n}{n}\,.$$

Lemma 5.2.2. *For any $a \in \mathbb{Q}_p$, $\log([\epsilon^a]) = a\log[\epsilon]$. Hence G_K acts on $t \cdot \mathbb{Q}_p$ by the cyclotomic character χ.*

Proof. It is formal that for $a \in \mathbb{Q}$, $\log([\epsilon^a]) = a \log[\epsilon]$.

We would like to argue that the equality holds for $a \in \mathbb{Q}_p$ by continuity. But the G_K-action on B_{dR}^+ is not jointly continuous for the $\ker \Theta_{\mathbb{Q}}$-adic topology, so we need to find a "better" topology.

Let ξ be a generator of $\ker \Theta$. Using Lemma 3.3.5 and the fact that the G_K-action on $\mathcal{O}_{C^\flat}$ is jointly continuous, we can verify that the G_K-action on A_{inf} is jointly continuous for the (p,ξ)-adic topology on A_{inf}. Give A_{inf}/ξ^n the quotient topology. Extend this topology to $A_{\mathrm{inf}}[1/p]/\xi^n$ by letting A_{inf}/ξ^n be open. Finally, give $B_{\mathrm{dR}}^+ = \varprojlim_n A_{\mathrm{inf}}[1/p]/\xi^n$ the inverse limit topology. Then G_K acts jointly continuously for this topology, and log is continuous on the open set $1 + \mathfrak{m}_{A_{\mathrm{inf}}} + \xi B_{\mathrm{dR}}^+ \subset B_{\mathrm{dR}}^+$, where $\mathfrak{m}_{A_{\mathrm{inf}}}$ is the maximal ideal of A_{inf}. $\square$

Lemma 5.2.3. *t is a uniformizer of B_{dR}^+.*

Proof. It is clear that $t \in \mathrm{Fil}^1 B_{\mathrm{dR}}^+$, so we just need to check that $t \notin \mathrm{Fil}^2 B_{\mathrm{dR}}^+$. For this, it is enough to check that $[\epsilon] - 1 \notin \mathrm{Fil}^2 B_{\mathrm{dR}}^+$, or equivalently, $[\epsilon] - 1 \notin (\ker \Theta)^2$.

For simplicity, we will assume $p \neq 2$. See [BC, Proposition 4.4.8] for the case $p = 2$.

Recall from the proof of Proposition 4.1.2 that $\ker \Theta \subset (p, [\pi^\flat])$, where $\pi^\flat \in \mathcal{O}_{C^\flat}$ satisfies $|\pi^\flat| = |p|$. So it is enough to check that $[\epsilon] - 1 \notin (p, [\pi^\flat]^2)$, i.e. $|\epsilon - 1| > |p|^2$.

Recall that if ζ_{p^n} is a primitive nth root of unity, then $|\zeta_{p^n} - 1| = |p|^{1/p^{n-1}(p-1)}$. Therefore,

$$|\epsilon - 1| = \lim_{n\to\infty} |\zeta_{p_n} - 1|^{p^n} = |p|^{p/(p-1)} > |p|^2 .$$

$\square$

Proposition 5.2.4. *There is a natural Galois-equivariant inclusion $\overline{K} \hookrightarrow B_{\mathrm{dR}}$.*

Proof. Let $\overline{k}$ be the residue field of $\overline{K}$. There is a natural inclusion $\overline{k} \hookrightarrow \mathcal{O}_{C^\flat}$ sending $x \mapsto \left([x^{p^{-n}}]\right)$, which induces inclusions $W(\overline{k}) \hookrightarrow A_{\mathrm{inf}}$, $W(\overline{k})[1/p] \hookrightarrow B_{\mathrm{dR}}^+$. Any $x \in \overline{K}$ satisfies an irreducible monic polynomial over $W(\overline{k})[1/p]$. This polynomial splits completely in C, the residue field of B_{dR}^+, so it also splits in B_{dR}^+ by Hensel's lemma. So there is a unique inclusion $\overline{K} \hookrightarrow B_{\mathrm{dR}}^+$ that makes the following diagram commute.

$$\begin{array}{ccc} W(\overline{k})[1/p] & \hookrightarrow & B_{\mathrm{dR}}^+ \\ \downarrow & \nearrow & \downarrow \\ \overline{K} & \hookrightarrow & C \end{array}$$

$\square$

Proposition 5.2.5. *There is a natural isomorphism $K \cong (B_{\mathrm{dR}}^+)^{G_K} = B_{\mathrm{dR}}^{G_K}$.*

Proof. By Proposition 5.2.4, we get an inclusion $K \hookrightarrow (B_{\mathrm{dR}}^+)^{G_K}$.

Now we show that the map $K \hookrightarrow (B_{\mathrm{dR}}^+)^{G_K}$ is also surjective and that $(B_{\mathrm{dR}}^+)^{G_K} = B_{\mathrm{dR}}^{G_K}$. For any n, by Lemma 5.2.2, we have an exact sequence

$$0 \to \mathrm{Fil}^{n+1} B_{\mathrm{dR}} \to \mathrm{Fil}^n B_{\mathrm{dR}} \to C(n) \to 0 .$$

Here we write $C(n)$ for $C(\chi^n)$. It induces an exact sequence

$$0 \to (\mathrm{Fil}^{n+1} B_{\mathrm{dR}})^{G_K} \to (\mathrm{Fil}^n B_{\mathrm{dR}})^{G_K} \to C(n)^{G_K} .$$

By Theorem 4.2.1, $C^{G_K} = K$ and $C(n)^{G_K} = 0$ if $n \neq 0$. Then $(\operatorname{Fil}^1 B_{\mathrm{dR}})^{G_K} = 0$, $B_{\mathrm{dR}}^{G_K} = (B_{\mathrm{dR}}^+)^{G_K}$, and the map

$$(B_{\mathrm{dR}}^+)^{G_K} \to C^{G_K} = K$$

is injective. On the other hand, the composition $K \to (B_{\mathrm{dR}}^+)^{G_K} \to K$ is the identity, so $(B_{\mathrm{dR}}^+)^{G_K} \to K$ must also be surjective. □

Definition 5.2.6. Let V be a finite-dimensional $\mathbb{Q}_p$-representation V of G_K. Define $D_{\mathrm{dR}}(V)$ to be the filtered K-vector space $(V \otimes_{\mathbb{Q}_p} B_{\mathrm{dR}})^{G_K}$.

We say that V is *de Rham* if $\dim_K D_{\mathrm{dR}}(V) = \dim_{\mathbb{Q}_p} V$.

If V is de Rham, then the *Hodge–Tate weights* of V are the integers i such that $\operatorname{gr}^i D_{\mathrm{dR}}(V) \neq 0$.

Example 5.2.7. The Hodge-Tate weight of $\mathbb{Q}_p(n) = \mathbb{Q}_p(\chi^n)$ is $-n$.

Theorem 5.2.8. *If X is a proper smooth variety over K, then $H^i_{\text{ét}}(X_{\overline{K}}, \mathbb{Q}_p)$ is de Rham, with Hodge–Tate weights between* 0 *and i, inclusive.*

This is proved as part of the de Rham comparison theorem.

Lemma 5.2.9. *Let L be a finite extension of K. Then a representation of G_K is de Rham if and only if its restriction to G_L is de Rham.*

Proof. This follows from Galois descent. □

Lemma 5.2.10. *A tensor product of two de Rham representations is de Rham.*

A subquotient of a de Rham representation is de Rham.

Proof. See [BC, §6.3]. □

Example 5.2.11. Let $\psi\colon \mathbb{Z}_p^\times \to \mathbb{Z}_p^\times$ be a character. The character $\psi \circ \chi\colon G_K \to \mathbb{Z}_p^\times$ is de Rham if and only if ψ is a product of a finite order character and a character of the form $z \mapsto z^n$ for some $n \in \mathbb{Z}$. In particular, there exist characters that are not de Rham. The "only if" direction follows from Theorem 4.2.1 by the same argument as in Proposition 5.2.5. For the "if" direction, we can use Lemma 5.2.9 to reduce to the case where the finite order character is trivial, and then apply Lemma 5.2.2.

6. B_{dR} AND DIFFERENTIALS

6.1. B_{dR} **and differentials.** It is not immediately obvious from the definition of B_{dR} that it should have anything to do with integrals of differential forms. We will now give an alternate characterization of B_{dR}^+ that suggests a connection to differential forms.

Let k be the residue field of K. Let

$$A_{\mathrm{inf},K} := A_{\mathrm{inf}} \otimes_{W(k)} \mathcal{O}_K\,.$$

There is a map $\Theta_K\colon A_{\mathrm{inf},K} \to \mathcal{O}_C$, and for each positive integer n, $B_{\mathrm{dR}}^+/\operatorname{Fil}^n B_{\mathrm{dR}} \cong A_{\mathrm{inf},K}/(\ker \Theta_K)^n[1/p]$. So

$$B_{\mathrm{dR}}^+ \cong \varprojlim_n \left(A_{\mathrm{inf},K}/(\ker \Theta_K)^n[1/p]\right)\,.$$

Inductively define

$$\mathcal{O}_{\overline{K}}^{(0)} := \mathcal{O}_{\overline{K}}$$

$$\Omega^{(n)} := \mathcal{O}_{\overline{K}} \otimes_{\mathcal{O}_{\overline{K}}^{(n-1)}} \Omega_{\mathcal{O}_{\overline{K}}^{(n-1)}/\mathcal{O}_K}$$

$$\mathcal{O}_{\overline{K}}^{(n)} := \ker\left(d^{(n)} \colon \mathcal{O}_{\overline{K}}^{(n-1)} \to \Omega^{(n)}\right)$$

Theorem 6.1.1.

(1) *For any nonnegative integer n, the preimage of $A_{\mathrm{inf},K}/(\ker\Theta_K)^{n+1}$ under the inclusion $\overline{K} \hookrightarrow B_{\mathrm{dR}}^+/\operatorname{Fil}^{n+1} B_{\mathrm{dR}}$ is $\mathcal{O}_{\overline{K}}^{(n)}$.*

(2) *For any nonnegative integers m, n, the map $\mathcal{O}_{\overline{K}}^{(n)}/p^m \to A_{\mathrm{inf},K}/((\ker\Theta_K)^{n+1}, p^m)$ is an isomorphism.*

(3) *B_{dR}^+ is the completion of $\overline{K}$ for the topology defined by letting the sets $p^m\mathcal{O}_{\overline{K}}^{(n)}$ for nonnegative integers m, n be a basis of open neighborhoods of the identity.*

(4) *For any nonnegative integer n, $d^{(n)}$ is surjective.*

Corollary 6.1.2. *The inclusion $\overline{K} \hookrightarrow B_{\mathrm{dR}}^+$ cannot be extended to a continuous map $C \to B_{\mathrm{dR}}^+$.*

Proof. Since $\overline{K}$ is dense in B_{dR}^+ and the projection $B_{\mathrm{dR}}^+ \to C$ is continuous and not injective, the topology on $\overline{K}$ considered as a subspace of B_{dR} must be finer than the topology on $\overline{K}$ considered as a subspace of C. □

Lemma 6.1.3. *The image of $\mathcal{O}_{\overline{K}}^{(n)}$ in B_{dR}^+ is contained in $A_{\mathrm{inf},K} + \operatorname{Fil}^{n+1} B_{\mathrm{dR}}$.*

Proof. We will use induction on n. The case $n = 0$ follows from the surjectivity of Θ.

Let $x \in \mathcal{O}_{\overline{K}}^{(n-1)}$. By the induction hypothesis, the image of x in B_{dR}^+ can be written (non-uniquely) as $x_0 + \epsilon$, with $x_0 \in A_{\mathrm{inf}}$ and $\epsilon \in \operatorname{Fil}^n B_{\mathrm{dR}}$. Consider the map

$$\mathcal{O}_{\overline{K}}^{(n-1)} \to \operatorname{Fil}^n B_{\mathrm{dR}}/\left((\ker\Theta_K)^n + \operatorname{Fil}^{n+1} B_{\mathrm{dR}}\right)$$

$$x \mapsto \epsilon$$

This map is an $\mathcal{O}_K$-linear derivation taking values in a $\mathcal{O}_{\overline{K}}$-module. By the universal property of $\Omega^{(n)}$, the map factors through $d^{(n)}$. In particular, its kernel contains $\ker d^{(n)} = \mathcal{O}_{\overline{K}}^{(n)}$. □

Lemma 6.1.4. *Let $x \in \mathcal{O}_{\overline{K}}$. Let $P \in \mathcal{O}_K[X]$ be a polynomial satisfying $P(x) = 0$, and let r be a nonnegative integer such that $p^r \mid P'(x)$ in $\mathcal{O}_{\overline{K}}$. For each nonnegative integer n, let $a_n = (2^n - 1)r$, $b_n = (2^{n+1} - 2n - 1)r$. Then for any positive integer m, $p^{a_n} x^m \in \mathcal{O}_{\overline{K}}^{(n)}$, and $x^{p^{b_n}} \in \mathcal{O}_{\overline{K}}^{(n)}$.*

Proof. Use induction on n. The base case $n = 0$ is trivial. Now assume that for some fixed n and all m, $p^{a_n} x^m \in \mathcal{O}_{\overline{K}}^{(n)}$, and that $x^{p^{b_n}} \in \mathcal{O}_{\overline{K}}^{(n)}$. By repeated use of the product rule, we see that

$$d^{(n+1)}(p^{2a_n} x^m) = m p^{a_n} x^{m-1} d^{(n+1)}(p^{a_n} x) \tag{6.1.5}$$

for each m. In particular, this implies

$$0 = d^{(n+1)}(p^{2a_n} P(x)) = p^{a_n} P'(x) d^{(n+1)}(p^{a_n} x)\,.$$

Hence

$$0 = d^{(n+1)}(p^{2a_n + r} x)\,. \tag{6.1.6}$$

Then multiplying (6.1.5) by p^r and applying (6.1.6) yields

$$0 = d^{(n+1)}(p^{2a_n + r} x^m) = d^{(n+1)}(p^{a_{n+1}} x^m)\,.$$

In the case $m = p^{b_n}$, since $p^r \mid m$, we get the stronger result

$$0 = d^{(n+1)}\left(p^{2a_n} x^{p^{b_n}}\right) = p^{2a_n} d^{(n+1)}\left(x^{p^{b_n}}\right)\,.$$

Then

$$d^{(n+1)}\left(x^{p^{b_{n+1}}}\right) = d^{(n+1)}\left(x^{p^{b_n + 2a_n}}\right) = p^{2a_n}(x^{p^{b_n}})^{p^{2a_n}-1} d^{(n+1)}\left(x^{p^{b_n}}\right) = 0\,.$$

□

Lemma 6.1.7. *$\mathcal{O}_{\overline{K}}^{(n)}$ is dense in $A_{\mathrm{inf}}/(\ker \Theta_K)^n$.*

Proof. Let π be a uniformizer of K. We claim that it suffices to check that $\mathcal{O}_{\overline{K}}^{(n)} \to \mathcal{O}_C/\pi = \mathcal{O}_{\overline{K}}/\pi$ is surjective. Indeed, for any integer m, $A_{\mathrm{inf}}/(\pi^m, (\ker \Theta_k)^n)$ is generated as an $\mathcal{O}_K$-module by the elements $[x]$ for $x \in \mathcal{O}_{K^\flat}$, and there is an integer r (independent of x) so that $([x^{p^{-r}}] + (\pi, \ker \Theta_K))^{p^r} \subseteq [x] + (\pi^m, (\ker \Theta_K)^n)$.

Now let $x \in \mathcal{O}_{\overline{K}}$, and let P be the minimal polynomial for x over k. Let x_m satisfy $x_m^{p^m} + \pi x_m = x$. Then $x_m^{p^m} \equiv x \pmod{\pi}$. We claim that for sufficiently large m, $x_m^{p^m} \in \mathcal{O}_{\overline{K}}^{(n)}$. Indeed, let $P_m(X) = P(X^{p^m} + \pi X)$; then $P_m(x_m) = 0$ and $P_m'(x_m) = (p^m x_m^{p^m - 1} + \pi) P'(x)$, so $|P_m'(x_m)| = |\pi||P'(x)|$. Then the claim follows from Lemma 6.1.4. □

Proof of Theorem 6.1.1(2). First, we prove item (2), that

$$\mathcal{O}_{\overline{K}}^{(n)}/p^m \to A_{\mathrm{inf}}/(p^m, (\ker \Theta)^{n+1})$$

is an isomorphism. Denote this map by $f_{m,n}$. We will construct an inverse map $g_{m,n}$. First, we need to prove the following claim.

Consider the map

$$\theta_{m,n} \colon \mathcal{O}_{\overline{K}}^{(n)}/p^m \to \mathcal{O}_C/p^m\,.$$

We will show by induction on n that $(\ker \theta_{m,n})^{n+1} = 0$. The base case $n = 0$ is trivial. Assume $(\ker \theta_{m,n-1})^n = 0$. It suffices to show that for $x \in \ker \theta_{m,n}$, $y \in (\ker \theta_{m,n})^n$, $xy = 0$. Choose lifts $\tilde{x}, \tilde{y} \in \mathcal{O}_{\overline{K}}^{(n)}$. By the induction hypothesis, $\tilde{y} \in p^m \mathcal{O}_{\overline{K}}^{(n-1)}$. Then $\tilde{x} \cdot p^{-m}\tilde{y} \in \mathcal{O}_{\overline{K}}^{(n-1)}$ and

$$d^{(n)}(\tilde{x} \cdot p^{-m}\tilde{y}) = p^{-m}\tilde{y} \cdot d^{(n)}\tilde{x} + p^{-m}\tilde{x} \cdot d^{(n)}\tilde{y} = 0\,.$$

So $\tilde{x}\tilde{y}$ is a multiple of p^m in $\mathcal{O}_{\overline{K}}^{(n)}$, implying $xy = 0$.

Now that we have proved the claim, we construct

$$g_{m,n} \colon A_{\mathrm{inf}}/(p^m, (\ker \Theta)^{n+1}) \to \mathcal{O}_{\overline{K}}^{(n)}/p^m$$

by sending $[x] \mapsto \tilde{x}^{p^{n+m+1}}$, where $\tilde{x}$ is some lift of $x^{(n+m+1)} + p^m \mathcal{O}_C$. It is easy to see that $f_{m,n} \circ g_{m,n} = 1$.

Now we prove that $g_{m,n} \circ f_{m,n} = 1$. Since $\mathcal{O}_{\overline{K}}^{(n)}$ has no p-torsion, $\widehat{\mathcal{O}_{\overline{K}}^{(n)}}$ also has no p-torsion. So it suffices to show that the induced maps

$$f_n \colon \widehat{\mathcal{O}_{\overline{K}}^{(n)}}[1/p] \to B_{\mathrm{dR}}^+/\operatorname{Fil}^{n+1} B_{\mathrm{dR}}^+$$

$$g_n \colon B_{\mathrm{dR}}^+/\operatorname{Fil}^{n+1} B_{\mathrm{dR}}^+ \to \widehat{\mathcal{O}_{\overline{K}}^{(n)}}[1/p]$$

satisfy $g_n \circ f_n = 1$. We can construct a map $\overline{K} \hookrightarrow \widehat{\mathcal{O}_{\overline{K}}}^{(n)}[1/p]$ by the same method as for $\overline{K} \to B_{\mathrm{dR}}$. It is not hard to see that $g_n \circ f_n$ fixes K, so $g_n \circ f_n$ must send $\overline{K}$ to itself. Since $g_n \circ f_n$ induces the identity on the residue field C, it must fix $\overline{K}$. But $\overline{K}$ is dense in $\widehat{\mathcal{O}_{\overline{K}}}^{(n)}[1/p]$, so $g_n \circ f_n = 1$. This concludes the proof of item (2). □

7. B_{dR} and differentials, Hodge–Tate decomposition for abelian varieties, p-divisible groups

7.1. B_{dR} and differentials, continued.

Proof of Theroem 6.1.1(1,3,4). Next, we prove item (1), that the preimage of $A_{\mathrm{inf},K}/(\ker\Theta_K)^{n+1}$ under the inclusion $\overline{K} \hookrightarrow B^+_{\mathrm{dR}}/\operatorname{Fil}^{n+1} B_{\mathrm{dR}}$ is $\mathcal{O}^{(n)}_{\overline{K}}$. Denote the preimage by $\mathcal{O}'$.

Let $x \in \mathcal{O}'$. Then by Lemma 6.1.4, there exists m so that $p^m x \in \mathcal{O}^{(n)}_{\overline{K}}$. By item (2), $\mathcal{O}^{(n)}_{\overline{K}}/p^m \to A_{\mathrm{inf}}/(p^m, (\ker\Theta_K)^{(n+1)})$ is injective, so $p^m x$ must be a multiple of p^m in $\mathcal{O}^{(n)}_{\overline{K}}$ as well. Hence $x \in \mathcal{O}^{(n)}_{\overline{K}}$.

Item (3) follows immediately from items (1) and (2).

Finally, we prove item (4). Each element of $\Omega^{(n)}$ is of the form $xd^{(n)}y$ for $x \in \mathcal{O}_{\overline{K}}$ and $y \in \mathcal{O}^{(n-1)}_{\overline{K}}$. By Lemma 6.1.4, we can find m so that $p^m d^{(n)}y = 0$, and by Lemma 6.1.7, we can find $z \in \mathcal{O}^{(n)}_{\overline{K}}$ so that $x - z \in p^m\mathcal{O}_{\overline{K}}$. Then $xd^{(n)}y = zd^{(n)}y = d^{(n)}(yz)$. □

Corollary 7.1.1. *For each positive integer n, there is a natural isomorphism*

$$\operatorname{Hom}_{\mathbb{Z}_p}(\mathbb{Q}_p, \Omega^{(n)}) \cong \operatorname{Fil}^n B_{\mathrm{dR}}/\operatorname{Fil}^{n+1} B_{\mathrm{dR}}\,.$$

Proof. By Theorem 6.1.1(4), there is an exact sequence

$$0 \to \mathcal{O}^{(n)}_{\overline{K}} \to \mathcal{O}^{(n-1)}_{\overline{K}} \to \Omega^{(n)} \to 0\,.$$

Recall that the functor $\operatorname{Ext}^1_{\mathbb{Z}}(\mathbb{Q}_p/\mathbb{Z}_p, -)$ is p-adic completion. So we get an exact sequence

$$0 \to \operatorname{Hom}_{\mathbb{Z}}(\mathbb{Q}_p/\mathbb{Z}_p, \Omega^{(n)}) \to A_{\mathrm{inf},K}/(\ker\Theta_K)^{n+1} \to A_{\mathrm{inf},K}/(\ker\Theta_K)^{n} \to 0\,.$$

Observe that $\operatorname{Hom}_{\mathbb{Z}}(\mathbb{Q}_p/\mathbb{Z}_p, \Omega^{(n)}) \cong \operatorname{Hom}_{\mathbb{Z}_p}(\mathbb{Q}_p/\mathbb{Z}_p, \Omega^{(n)})$, and since $\Omega^{(n)}$ is p-power torsion, $\operatorname{Hom}_{\mathbb{Z}_p}(\mathbb{Q}_p/\mathbb{Z}_p, \Omega^{(n)}) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p \cong \operatorname{Hom}_{\mathbb{Z}_p}(\mathbb{Q}_p, \Omega^{(n)})$. Then inverting p gives an exact sequence

$$0 \to \operatorname{Hom}_{\mathbb{Z}_p}(\mathbb{Q}_p, \Omega^{(n)}) \to B^+_{\mathrm{dR}}/\operatorname{Fil}^{n+1} B^+_{\mathrm{dR}} \to B^+_{\mathrm{dR}}/\operatorname{Fil}^{n} B^+_{\mathrm{dR}} \to 0\,.$$

□

7.2. Hodge-Tate decomposition for abelian varieties.

Recall the comparison isomorphism

$$H^n_{\mathrm{dR}}(X/K) \otimes_K B_{\mathrm{dR}} \cong H^n_{\text{ét}}(X_{\overline{K}}, \mathbb{Z}_p) \otimes_{\mathbb{Z}_p} B_{\mathrm{dR}}\,.$$

This is an isomorphism of filtered vector spaces with G_K-action. Taking the zeroth graded piece gives an isomorphism

$$\bigoplus_{i=0}^{n} H^{n-i}(X, \Omega^i_{X/K}) \otimes_K C(-i) \cong H^n_{\text{ét}}(X_{\overline{K}}, \mathbb{Z}_p) \otimes_{\mathbb{Z}_p} C\,.$$

In the case where X is an abelian variety, this decomposition can actually be made fairly explicit. In that case, all cohomology groups are wedge powers of H^1, so it suffices to consider $n = 1$. Recall that $H^1_{\text{ét}}(X_{\overline{K}}, \mathbb{Z}_p)$ is dual to the Tate module $T_p(X) := \varprojlim_n X(\overline{K})[p^n]$. Therefore, giving a C-linear map $H^0(X, \Omega^1_{X/K}) \otimes_K C(-1) \to H^1_{\text{ét}}(X_{\overline{K}}, \mathbb{Z}_p) \otimes_{\mathbb{Z}_p} C$ is equivalent to giving a map

$$H^0(X, \Omega^1_{X/K}) \times T_p(X) \to C(1)$$

that is K-linear in the first variable and $\mathbb{Z}_p$-linear in the second.

We will sketch the construction of the map, but we refer the reader to [Fon82] for the technical details.

Let $\mathfrak{X}$ be a proper flat model of X over $\mathcal{O}_K$. We can define a map

$$H^0(\mathfrak{X}, \Omega^1_{\mathfrak{X}/\mathcal{O}_K}) \times \mathfrak{X}(\mathcal{O}_{\overline{K}}) \to \Omega^{(1)}$$
$$(\omega, x) \mapsto x^*(\omega)\,.$$

It is possible to use the group law on the generic fiber show that for some r, the restriction

$$p^r H^0(\mathfrak{X}, \Omega^1_{\mathfrak{X}/\mathcal{O}_K}) \times \mathfrak{X}(\mathcal{O}_{\overline{K}}) \to \Omega^{(1)}$$

is bilinear.

By the valuative criterion of properness, $\mathfrak{X}(\mathcal{O}_{\overline{K}}) = X(K)$. Using $\mathrm{Hom}(\mathbb{Q}_p/\mathbb{Z}_p, X(K)) \cong T_p(X)$ and $\mathrm{Hom}(\mathbb{Q}_p/\mathbb{Z}_p, \Omega^{(1)}) \cong \ker\Theta_K/(\ker\Theta_K)^2$, we obtain a map

$$p^r H^0(\mathfrak{X}, \Omega^1_{\mathfrak{X}/\mathcal{O}_K}) \times T_p(X) \to (\ker\Theta_K)/(\ker\Theta_K)^2\,.$$

Finally, we invert p to get a bilinear map

$$H^0(X, \Omega^1_{X/K}) \times T_p(X) \to C(1)\,.$$

Now we have a constructed map (which can be shown to be injective)

$$H^0(X, \Omega^1_{X/K}) \otimes_K C(-1) \hookrightarrow H^1_{\text{ét}}(X_{\overline{K}}, \mathbb{Z}_p) \otimes_{\mathbb{Z}_p} C\,. \tag{7.2.1}$$

If $X^\vee$ is the dual abelian variety, then we get a map

$$H^0(X^\vee, \Omega^1_{X^\vee/K}) \otimes_K C(-1) \hookrightarrow H^1_{\text{ét}}(X^\vee_{\overline{K}}, \mathbb{Z}_p) \otimes_{\mathbb{Z}_p} C\,.$$

The Weil pairing determines a perfect pairing

$$H^1_{\text{ét}}(X_{\overline{K}}, \mathbb{Z}_p) \times H^1_{\text{ét}}(X^\vee_{\overline{K}}, \mathbb{Z}_p) \to \mathbb{Z}_p(-1)$$

and there is also a perfect pairing

$$H^1(X, \mathcal{O}_X) \times H^0(X^\vee, \Omega^1_{X^\vee/K}) \to K\,.$$

So we get a map

$$H^1(X, \mathcal{O}_X)^* \otimes_K C(-1) \hookrightarrow H^1_{\text{ét}}(X_{\overline{K}}, \mathbb{Z}_p)^* \otimes_{\mathbb{Z}_p} C(-1)\,.$$

Taking the dual and twisting, we get a map

$$H^1_{\text{ét}}(X_{\overline{K}}, \mathbb{Z}_p) \otimes_{\mathbb{Z}_p} C \twoheadrightarrow H^1(X, \mathcal{O}_X) \otimes_K C\,. \tag{7.2.2}$$

The composite of (7.2.1) and (7.2.2) must be zero since $C(1)^{G_K} = 0$. Then, by dimension counting, the sequence

(7.2.3)

$$0 \to H^0(X, \Omega^1_{X/K}) \otimes_K C(-1) \to H^1_{\text{ét}}(X_{\overline{K}}, \mathbb{Z}_p) \otimes_{\mathbb{Z}_p} C \to H^1(X, \mathcal{O}_X) \otimes_K C \to 0$$

must be exact. Since $H^1(G_K, C(-1)) = 0$ (see Remark 4.2.2), this sequence has a G_K-equivariant splitting. Since $C(-1)^{G_K} = 0$, this splitting is unique.

subsection$p$-divisible groups Now we introduce p-divisible groups, another source of Galois representations.

Definition 7.2.4. Let S be a scheme. A *finite locally free group scheme over S* is a commutative group scheme G over S such that the pushforward of $\mathcal{O}_G$ is a finite locally free $\mathcal{O}_S$-module.

A *p-divisible group of height h over S* is an inductive system $G = (G_v, \iota_v), v \in \mathbb{N}$, where G_v is a finite locally free group scheme over S of order p^{vh} and $\iota_v \colon G_v \to G_{v+1}$ has the property that

$$0 \to G_v \xrightarrow{\iota_v} G_{v+1} \xrightarrow{p^v} G_{v+1}$$

is exact.

Remark 7.2.5. In the literature, the term "finite flat group scheme" is often used. If S is locally noetherian, then there is no difference between these two notions, but in general, finite locally free group schemes are better behaved. See [Sta, Tag 02K9].

Example 7.2.6. Some examples of finite locally free group schemes are:

- If G is a finite abelian group, then the functor $\underline{G}$ that sends a scheme X to the set of continuous maps $|X| \to G$ is representable by a finite locally free group scheme over $\mathbb{Z}$. As a scheme, it is a disjoint union of $|G|$ copies of $\operatorname{Spec} \mathbb{Z}$.
- $\mu(N)$, the group of Nth roots of unity, is a finite locally free group scheme over $\mathbb{Z}$. Its Hopf algebra is $\mathbb{Z}[X]/(X^N - 1)$, with coproduct $X \mapsto X \otimes X$.
- α_p is a finite locally free group scheme over $\mathbb{F}_p$. Its Hopf algebra is $\mathbb{F}_p[X]/(X^p)$, with coproduct $X \mapsto X \otimes 1 + 1 \otimes X$.
- If A is a semiabelian variety over a field K and N is a positive integer, then the N-torsion subgroup $A[N]$ is a finite locally free group scheme.

Example 7.2.7. Some examples of p-divisible groups are:

- $\underline{\mathbb{Q}_p/\mathbb{Z}_p}$ is a p-divisible group over $\mathbb{Z}$ of height 1 ($G_v = \underline{p^{-v}\mathbb{Z}/\mathbb{Z}}$)
- $\mu(p^\infty)$ is a p-divisible group over $\mathbb{Z}$ of height 1 ($G_v = \mu(p^v) = \operatorname{Spec} \mathbb{Z}[T]/((1+T)^{p^v} - 1)$)
- If A is an semiabelian variety over a field K, then $A[p^\infty]$ is a p-divisible group over S ($G_v = A[p^v]$). If A is an abelian variety, then the height of $A[p^\infty]$ is $2 \dim A$. If A is a torus, then the height of $A[p^\infty]$ is $\dim A$.

8. More on p-divisible groups

8.1. Constructions involving p-divisible groups.

Definition 8.1.1. Let G be a finite locally free group scheme over a scheme S. Then the *Cartier dual* of G, denoted $G^\vee$, is the finite locally free group scheme representing the functor from schemes over S to sets given by $T \mapsto \operatorname{Hom}(G_T, (\mathbb{G}_m)_T)$. Here Hom is taken in the category of group schemes.

The *Cartier dual* of a p-divisible group $G = (G_v)$ is $G^\vee := (G_v^\vee)$. It represents the functor from schemes over S to sets given by $T \mapsto \operatorname{Hom}(G_T, \mu(p^\infty))$. Here Hom is taken in the category of p-divisible groups.

The Hopf algebra of $G^\vee$ is dual to the Hopf algebra of G. Let $f \colon G \to S$ be the projection map. Then there is a product $f_*\mathcal{O}_G \otimes_{\mathcal{O}_S} f_*\mathcal{O}_G \to f_*\mathcal{O}_G$ induced by multiplication on $\mathcal{O}_G$ and a coproduct $f_*\mathcal{O}_G \to f_*\mathcal{O}_G \otimes_{\mathcal{O}_S} f_*\mathcal{O}_G$ induced by the group operation $G \times G \to G$. Taking the duals of these maps gives us a coproduct and product, respectively, on $(f_*\mathcal{O}_G)^\vee$, and we can take $G^\vee = \operatorname{Spec}((f_*\mathcal{O}_G)^\vee)$.

From the above description, we see that for any finite locally free group scheme or p-divisible group G, $(G^\vee)^\vee$ is naturally isomorphic to G.

If G is a finite locally free group scheme, then there is a canonical bilinear pairing $G \times G^\vee \to \mathbb{G}_m$. If G is a p-divisible group, then there is a canonical bilinear pairing $G \times G^\vee \to \mu_{p^\infty}$.

Example 8.1.2.

(1) $\underline{\mathbb{Z}/p^n\mathbb{Z}}$ and $\mu(p^n)$ are a Cartier dual pair of finite locally free group schemes.
(2) $\underline{\mathbb{Q}_p/\mathbb{Z}_p}$ and $\mu(p^\infty)$ are a Cartier dual pair of p-divisible groups.
(3) If A is an abelian variety over a field K and $A^\vee$ is the dual abelian variety, then $A[p^n]$ and $A^\vee[p^n]$ are Cartier duals, as are $A[p^\infty]$ and $A^\vee[p^\infty]$.
(4) α_p is its own Cartier dual. The bilinear pairing $\alpha_p \times \alpha_p \to \mathbb{G}_m$ is given by given by $(x, y) \mapsto \sum_{n=0}^{p-1} \frac{(xy)^n}{n!}$.

Definition 8.1.3. Let G be a p-divisible group over a field K. Define the *Tate module*

$$T(G) := \varprojlim_v G_v(K^{\text{sep}}) .$$

Example 8.1.4. If $G = \underline{\mathbb{Q}_p/\mathbb{Z}_p}$, then $T(G) \cong \mathbb{Z}_p$ with trivial Galois action. If $G = \mu(p^\infty)$, then $T(G) \cong \underline{\mathbb{Z}_p(1)}$.

Proposition 8.1.5. *Let G be a finite locally free group scheme over a Henselian local ring R. There is an exact sequence*

$$0 \to G^0 \to G \to G^{\text{ét}} \to 0$$

where G^0 is (geometrically) connected and $G^{\text{ét}}$ is étale over R.

A similar result holds for p-divisible groups.

Proof. See [Tat97, §3.7]. □

Remark 8.1.6. If p is invertible in R, then every finite locally free group scheme over R is étale over R.

Example 8.1.7. $\underline{\mathbb{Q}_p/\mathbb{Z}_p}$ is étale over $\mathbb{Z}$.

Suppose that R is a Henselian local ring with residue characteristic p. Then the p-divisible group μ_{p^∞} over R is connected. Let E be an elliptic curve over R. If the special fiber of E is supersingular, then $E[p^\infty]$ is connected. If the special fiber is ordinary, then $E[p^\infty]^0$ and $E[p^\infty]^{\text{ét}}$ each have height 1.

Theorem 8.1.8. *Let K be a nonarchimedean field of mixed characteristic. Let G be a p-divisible group over $\mathcal{O}_K$. Let $\operatorname{Nilp}_{\mathcal{O}_K}$ be the category of $\mathcal{O}_K$-algebras on which p is nilpotent. The contravariant functor $\operatorname{Nilp}_{\mathcal{O}_K} \to \mathrm{Set}$ that sends $R \mapsto \varinjlim_v G_v(R)$ is representable by a formal scheme $\mathfrak{G}$ over $\operatorname{Spf} \mathcal{O}_K$.*

Proof. Combine Proposition 8.1.5 with [SW13, Lemma 3.1.1]. The connected case is proved in [Mes72, Theorem II.2.1.8]. □

If R is a p-adically complete and separated $\mathcal{O}_K$-algebra, then we will abuse notation and write $G(R)$ for

$$\mathfrak{G}(R) = \varprojlim_n \varinjlim_v G_v(R/p^nR) .$$

Example 8.1.9. The formal scheme associated with $\mu(p^\infty)$ is is the formal multiplicative group $\widehat{\mathbb{G}}_m = \operatorname{Spf} \mathcal{O}_K [\![T]\!]$. (Here $1+T$ is the coordinate on the open unit disc of radius 1 centered at 1). Somewhat counterintuitively, $\mu(p^\infty)(\mathcal{O}_C) = (1+\mathfrak{m}_C)^\times$, not the group of roots of unity of $\mathcal{O}_C$. This is because any $x \in \mathfrak{m}_C$ has the property that $(1+x)^{p^n} \to 1$ as $n \to \infty$.

Theorem 8.1.10.

(1) *Let*
$$\operatorname{Lie} G := \ker\left(G(\mathcal{O}_K[\epsilon]/(\epsilon^2)) \to G(\mathcal{O}_K)\right)$$
be the tangent space to $\mathfrak{G}$ *at the origin. Then* $\operatorname{Lie} G$ *is a finite free* $\mathcal{O}_K$*-module.*

(2) *If* G *is connected, then* $\mathfrak{G} \cong \operatorname{Spf} \mathcal{O}_K [\![X_1, \ldots, X_n]\!]$*, where* n *is the rank of* $\operatorname{Lie} G$.

Proof. See [Mes72, Theorem II.2.1.8], or [Tat67, §2.2] in the case where K is discretely valued. □

Definition 8.1.11. The *dimension* of G is the rank of $\operatorname{Lie} G$.

8.2. **Hodge–Tate decomposition for p-divisible groups over $\mathcal{O}_C$.** Let C be an algebraically closed nonarchimedean field of mixed characteristic $(0,p)$, and let $\mathcal{O}_C$ be the ring of integers of C. Define
$$T(G) := T(G_C) = \varprojlim_v G_v(C)\,.$$

Recall that Cartier duality gives a pairing
$$G \times G^\vee \to \mu_{p^\infty}\,.$$

There are induced pairings
$$T(G) \times T(G^\vee) \to T(\mu_{p^\infty}) = \mathbb{Z}_p(1)$$
$$\operatorname{Lie} G \times T(G^\vee) \to \operatorname{Lie} \mu_{p^\infty} = \mathcal{O}_C$$
$$T(G) \times \operatorname{Lie} G^\vee \to \operatorname{Lie} \mu_{p^\infty} = \mathcal{O}_C$$

The first pairing is perfect, so we can identify $T(G^\vee) \cong T(G)^\vee(1)$. So we get maps
$$(\operatorname{Lie} G)(1) \to T(G) \otimes_{\mathbb{Z}_p} \mathcal{O}_C$$
$$T(G) \otimes_{\mathbb{Z}_p} \mathcal{O}_C \to (\operatorname{Lie} G^\vee)^\vee\,.$$
If we tensor with C, we get
$$\operatorname{Lie} G \otimes_{\mathcal{O}_C} C(1) \to T(G) \otimes_{\mathbb{Z}_p} C \tag{8.2.1}$$
$$T(G) \otimes_{\mathbb{Z}_p} C \to \operatorname{Lie}(G^\vee)^\vee \otimes_{\mathcal{O}_C} C\,. \tag{8.2.2}$$

Theorem 8.2.3. *The maps* (8.2.1) *and* (8.2.2) *induce an exact sequence*
$$0 \to \operatorname{Lie} G \otimes_C C(1) \to T(G) \otimes_{\mathbb{Z}_p} C \to (\operatorname{Lie} G^\vee)^\vee \otimes C \to 0\,. \tag{8.2.4}$$

Proof. See [Far08, Thm. II.1.1], or [Tat67, Thm. 3] for the case where K is discretely valued. □

If $G = A[p^\infty]$ for some abelian variety A over $\mathcal{O}_C$, then we can identify $H^0(A_C, \Omega^1_{A_C/C}) \cong (\operatorname{Lie} G)^\vee \otimes_{\mathcal{O}_C} C$, $H^1(A_C, \mathcal{O}_{A_C}) \cong \operatorname{Lie} G^\vee \otimes_{\mathcal{O}_C} C$, and the above exact sequence is dual to
$$0 \to H^1(A_C, \mathcal{O}_{A_C}) \to H^1_{\text{ét}}(A_C, \mathbb{Z}_p) \otimes_{\mathbb{Z}_p} C \to H^0(A_C, \Omega^1_{A/C}) \otimes_C C(-1) \to 0\,.$$

Remark 8.2.5. Note that the maps in this sequence are in the opposite direction to those in (7.2.3). In section 7.2, we assumed that A is defined over a p-adic field K, in which case the sequences have a unique G_K-equivariant splitting. But over C, there is not a canonical splitting.

Theorem 8.2.6. *Let K be a p-adic field, and let H_1 and H_2 be p-divisible groups over $\mathcal{O}_K$. Any homomorphism $(H_1)_K \to (H_2)_K$ extends uniquely to a homomorphism $H_1 \to H_2$.*

Remark 8.2.7. The assumption that K is a p-adic field (i.e. it is discretely valued) is necessary. If C is algebraically closed, then any two p-divisible groups over C of the same height are isomorphic, but the same is not true over $\mathcal{O}_C$.

Proof. We will just give a sketch; see [Tat67, Thm. 4] for details.

The first step in the proof is to show that if $H_1 \to H_2$ is a map of p-divisible groups over $\mathcal{O}_K$ that induces an isomorphism on the generic fibers, then the map itself is an isomorphism. For each v, let $(H_1)_v = \operatorname{Spec} R_v$, $(H_2)_v = \operatorname{Spec} S_v$. Then $S_v[1/p] \cong R_v[1/p]$, and $S_v \to R_v$ is an isomorphism iff S_v and R_v have the same discriminant over $\mathcal{O}_K$. One can show (without the discrete valuation assumption) that the discriminant depends only on the height and dimension of the group. We claim that the height and dimension of H_i can be recovered from $T(H_i)$, which depends only on the generic fiber of H_i. Indeed, $\operatorname{ht} H_i = \operatorname{rk} T(H_i)$ and the Hodge–Tate decomposition (8.2.4) implies $\dim H_i = \dim_K (T(H_i) \otimes C(-1))^{G_K}$. (Here we use that $C(-1)^{G_K} = 0$, which relies on the assumption that G_K is discretely valued.)

Suppose we have a homomorphism $f\colon (H_1)_K \to (H_2)_K$. Let Γ_K be the image of the graph morphism $\operatorname{id} \times f\colon (H_1)_K \to (H_1)_K \times (H_2)_K$, and let Γ be the closure of Γ_K in $H_1 \times H_2$. Then the projection $\Gamma \to H_1$ is an isomorphism since it induces an isomorphism on generic fibers. So composing the inverse $H_1 \to \Gamma$ with the projection $\Gamma \to H_2$ gives the desired map $H_1 \to H_2$. □

9. Classifications of p-divisible groups, crystalline representations

9.1. **Classifications of p-divisible groups.**

Proposition 9.1.1. *Let R be a Henselian local ring with residue field k. The functor $G \mapsto G(R^{\mathrm{sh}})$ induces an isomorphism between the category of étale finite locally free group schemes over R and the category of finitely generated torsion $\mathbb{Z}_p$-modules with continuous $\operatorname{Gal}(k^{\mathrm{sep}}/k)$-action.*

The functor T induces an equivalence between the category of étale p-divisible groups over R and the category of finite free $\mathbb{Z}_p$-modules with continuous $\operatorname{Gal}(k^{\mathrm{sep}}/k)$-action.

In characteristic p, Tate modules are less useful, since the Tate module of a connected p-divisible group is zero. However, over a perfect field k, Dieudonné modules provide a convenient way of describing p-divisible groups.

Definition 9.1.2. Let k be a perfect field. The *Dieudonné ring* D_k is the (noncommutative) ring over $W(k)$ generated by element F and V subject to the relation $FV = VF = p$, $Fc = \phi(c)F$, $cV = V\phi(c)$ for $c \in k$. Here ϕ is the Frobenius endomorphism of $W(k)$.

Theorem 9.1.3. *There is an anti-equivalence of categories $G \mapsto \mathbf{D}(G)$ between the category of finite locally free group schemes of p-power order over k and the category of left D_k-modules of finite $W(k)$-length.*

Theorem 9.1.4. *There is an anti-equivalence of categories $G \mapsto \mathbf{D}(G)$ between the category of p-divisible groups over k and the category of left D_k-modules that are finite free $W(k)$-modules.*

I will not say exactly how the functor is defined, but the basic idea is that $\mathbf{D}(G)$ is the set of maps from G to the scheme of "Witt co-vectors". If R is perfect, then the group of R-points of this scheme is isomorphic to the additive group $W(R)[1/p]/W(R)$.

Remark 9.1.5. The actions of F and V are induced by maps of group schemes. More specifically, there is a commutative diagram

$$\begin{array}{ccccc} & & \overset{F_{\text{abs}}}{\frown} & & \\ G & \xrightarrow{F} & G^{(p)} & \longrightarrow & G \\ & \searrow & \downarrow & & \downarrow \\ & & \operatorname{Spec} k & \xrightarrow{F_{\text{abs}}} & \operatorname{Spec} k \end{array} .$$

Here F_{abs} is the absolute Frobenius, F is the geometric Frobenius, and $G^{(p)}$ is defined to make the square Cartesian. Furthermore, since multiplication by p is completely inseparable, it factors through F. There is a commutative diagram

$$\begin{array}{ccccc} & & \overset{p}{\frown} & & \\ G & \xrightarrow{F} & G^{(p)} & \xrightarrow{V} & G \end{array} .$$

One can also define V to be the Cartier dual of $F\colon G^\vee \to (G^\vee)^{(p)}$.
Example 9.1.6.

(1) $\mathbf{D}(\underline{\mathbb{Z}/p^n\mathbb{Z}}) \cong W(k)/p^n$ and $\mathbf{D}(\underline{\mathbb{Q}_p/\mathbb{Z}_p}) \cong W(k)$, where $F[x] = [x^p]$ and $V[x] = p[x^{1/p}]$.
(2) $\mathbf{D}(\mu(p^n)) \cong W(k)/p^n$ and $\mathbf{D}(\mu_{p^\infty}) \cong W(k)$, where $F[x] = p[x^p]$ and $V[x] = [x^{1/p}]$.
(3) $\mathbf{D}(\alpha_p) = k$, where $F = V = 0$.
(4) If A is an abelian varety, then $V \cdot \mathbf{D}(A[p^\infty])$ can be identified with $H^1_{\text{cris}}(A/W(k))$.
(5) Let E be an elliptic curve, and let $M = \mathbf{D}(E[p^\infty])$. Then M is a free $W(k)$-module of rank 2. If E is ordinary, then $M/FM \cong W_2(k) = W(k)/p^2W(k)$. If E is supersingular, then $M/FM \cong k^2$.

Definition 9.1.7. A *Honda system* over $W(k)$ is a pair (M, L) consisting of a left D_k-module M and a $W(k)$-submodule L such that M is a finite free $W(k)$-module and the induced map $L/pL \to M/FM$ is an isomorphism.

A *finite Honda system* over $W(k)$ is a pair (M, L) consisting of a left D_k-module M and a $W(k)$-submodule L such that M has finite $W(k)$-length, the induced map $L/pL \to M/FM$ is an isomorphism, and $\ker V \cap L = 0$.

Theorem 9.1.8. *Let k be a perfect field of characteristic $p > 2$.*

There is a natural anti-equivalence of categories $G \mapsto (\mathbf{D}(G_k), L(G))$ between the category of p-divisible groups over $W(k)$ and the category of Honda systems.

There is a natural anti-equivalence of categories $G \mapsto (\mathbf{D}(G_k), L(G))$ between the category of finite locally free group schemes over $W(k)$ and the category of finite Honda systems.

Example 9.1.9.

(1) If G is étale, then $L = 0$.
(2) If $G^\vee$ is étale, then $L = \mathbf{D}(G_k)$.
(3) If A is an abelian variety over $W(k)$, and $M = \mathbf{D}(A_k[p^\infty])$, then we can identify $VM \cong H^1_{\mathrm{dR}}(A/W(k))$, and VL is the subspace corresponding to $H^0(A, \Omega^1_{A/W(k)})$.
(4) If E is an elliptic curve with supersingular reduction and $G = E[p^\infty]$, then L is one-dimensional, and it is generated by an element not in the image of F.

Theorem 9.1.10. *Let C be an algebraically closed nonarchimedean field of residue characteristic p, and let $\mathcal{O}_C$ be the ring of integers of C. There is an equivalence of categories between the category of p-divisible groups over $\mathcal{O}_C$ and the category of free $\mathbb{Z}_p$-modules T of finite rank together with a C-sub-vector space W of $T \otimes C(-1)$. The equivalence is characterized by*

$$T = T(G)$$

$$W = \mathrm{im}\left(\mathrm{Lie}\, G \otimes_{\mathcal{O}_C} C \to T(G) \otimes C(-1)\right) .$$

Proof. See [SW13, Theorem 5.2.1]. □

9.2. **The crystalline comparison theorem.** Let K be a p-adic field. Let $\mathcal{O}_K$ be the ring of integers of K, and let k be the residue field of K. Let $K_0 = W(k)[1/p]$.

Theorem 9.2.1. *Let X be a proper smooth scheme over $\mathcal{O}_K$. Then there is a G_K- and Frobenius- equivariant isomorphism*

$$H^*_{\text{ét}}(X_{\overline{K}}, \mathbb{Z}_p) \otimes_{\mathbb{Z}_p} B_{\mathrm{cris}} \cong H^*_{\mathrm{cris}}(X_k/W(k)) \otimes_{W(k)} B_{\mathrm{cris}} .$$

We will define B_{cris} later. It has the following properties.

- B_{cris} is a subring of B_{dR} (hence B_{cris} is an integral domain).
- $(B_{\mathrm{cris}})^{G_K} = (\mathrm{Frac}\, B_{\mathrm{cris}})^{G_K} = K_0$.
- $(B_{\mathrm{cris}} \otimes_{K_0} K)^{G_K} = (\mathrm{Frac}\, B_{\mathrm{cris}} \otimes_{K_0} K)^{G_K} = K$.
- $\breve{K}_0 := W(\overline{k})[1/p] \subset B_{\mathrm{cris}}$
- $t \in B_{\mathrm{cris}}$

Definition 9.2.2. Let k be a perfect field of characteristic p, and let $K_0 = W(k)[1/p]$. An *isocrystal* over K_0 is a finite-dimensional K_0-vector space D equipped with a bijective Frobenius-semilinear endomorphism $\phi_D \colon D \to D$.

Example 9.2.3.

(1) If X is a proper smooth scheme over k, then $H^i_{\mathrm{cris}}(X/W(k)) \otimes_{W(k)} K_0$ is an isocrystal over K_0.
(2) If G is a p-divisible group over k, then $\mathbf{D}(G) \otimes_{W(k)} K_0$ is an isocrystal over K_0.
(3) If V is a finite-dimensional representation of G_K, then $(V \otimes_{\mathbb{Q}_p} B_{\mathrm{cris}})^{G_K}$ is an isocrystal over K_0.

10. Filtered ϕ-modules and crystalline representations

10.1. Filtered ϕ-modules.

Definition 10.1.1. Let K be a p-adic field with residue field k, and let $K_0 = W(k)[1/p]$. A *filtered ϕ-module over K* is a pair $(D, \mathrm{Fil}^\bullet)$ consisting of an isocrystal D over K_0 and a decreasing exhaustive and separated filtration $\mathrm{Fil}^\bullet$ on $D_K := D \otimes_{K_0} K$.

Example 10.1.2.

(1) Let X is a proper smooth scheme over $\mathcal{O}_K$. Then there is an isomorphism

$$H^i_{\mathrm{cris}}(X_k/W(k)) \otimes_{W(k)} K \cong H^i_{\mathrm{dR}}(X_K/K)\,.$$

Then the pair consisting of $H^i_{\mathrm{cris}}(X_k/W(k)) \otimes_{W(k)} K_0$ and the Hodge filtration on $H^i_{\mathrm{dR}}(X_K/K)$ is a filtered ϕ-module over K.

(2) If V be a finite-dimensional $\mathbb{Q}_p$-representation of G_K, then let $D_{\mathrm{cris}}(V)$ be the pair consisting of $(V \otimes_{\mathbb{Q}_p} B_{\mathrm{cris}})^{G_K}$ and the filtration induced from $(V \otimes_{\mathbb{Q}_p} B_{\mathrm{cris}} \otimes_{K_0} K)^{G_K} \hookrightarrow D_{\mathrm{dR}}(V)$ is a filtered ϕ-module over K.

Definition 10.1.3. A finite-dimensional $\mathbb{Q}_p$-representation V of G_K is *crystalline* if $\dim_{K_0} D_{\mathrm{cris}}(V) = \dim_{\mathbb{Q}_p} V$.

Example 10.1.4. Let V be an unramified representation of G_K. Then we can also view V as a representation of G_k. Then V is crystalline and $D_{\mathrm{cris}}(V) \cong D_{\mathcal{E}}(V)$, where $D_{\mathcal{E}}$ is the functor of Theorem 2.1.11. In this case, the field $\mathcal{E}$ is K_0.

One might ask which filtered ϕ-modules come from crystalline representations. To answer this question, we will need to introduce Newton and Hodge polygons.

Theorem 10.1.5 (Dieudonné–Manin decomposition)**.** *Let k be an algebraically closed field of characteristic p, and let $K_0 = W(k)[1/p]$.*

Let r be a positive integer, and let s be an integer relatively prime to r. Define the isocrystal $D_{r,s}$ over K_0 to be $(K_0)^r$, with Frobenius action given by $\phi(e_i) = e_{i+1}$ for $1 \le i \le r$, $\phi(e_r) = p^s e_1$.

If $(r,s) \neq (r',s')$, then $\mathrm{Hom}(D_{r,s}, D_{r',s'}) = 0$. *Moreover, any isocrystal D over K_0 is isomorphic to a direct sum of copies of modules of the form $D_{r,s}$.*

Definition 10.1.6. Let k be a perfect field of characteristic p, let $K_0 = W(k)[1/p]$, and let $\check{K}_0 = W(\overline{k})(1/p)$. Let D be an isocrystal over K_0. The fractions $\frac{s}{r}$ appearing in the decomposition of $D \otimes_{K_0} \check{K}_0$ of Theorem 10.1.5 are called the *slopes* of D.

Lemma 10.1.7. *Let D be an isocrystal over $\mathbb{Q}_{p^n}$. Then the slopes of D are $v_p(\lambda)/n$, where λ runs over eigenvalues of ϕ^n.*

Proof. Factor the characteristic polynomial of ϕ^n into irreducibles; each irreducible determines a subspace of D on which all eigenvalues of ϕ^n have the same p-adic valuation. So we can reduce to the case where all eigenvalues have the same valuation.

For any (r,s), we can find $D_{r,s,\mathbb{Q}_p}$ over $\mathbb{Q}_p$ so that $D_{r,s,\mathbb{Q}_p} \otimes_{\mathbb{Q}_p} \check{\mathbb{Q}}_p \cong D_{r,s}$ and the eigenvalues of ϕ are the rth roots of p^s. By tensoring with $D_{r,s,\mathbb{Q}_p}$, we reduce to the case where the eigenvalues have valuation zero.

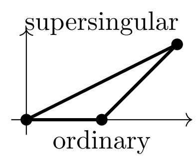
Now suppose that the eigenvalues of ϕ^n have valuation zero. Then D admits a lattice Λ such that $\phi\Lambda = \Lambda$, so $D \otimes_{\mathbb{Q}_{p^n}} \check{\mathbb{Q}}_p$ does as well. Then all slopes of D must be zero. □

Definition 10.1.8. Let k be a perfect field of characteristic p, let $K_0 = W(k)[1/p]$, and let $\breve{K}_0 = W(\overline{k})[1/p]$. Let D be an isocrystal over K_0. Suppose that $D \otimes_{K_0} \breve{K}_0 \cong \bigoplus_{i=1}^n D_{r_i,s_i}$ with the s_i/r_i in nondecreasing order. Then the *Newton polygon* $P_N(D)$ of D is the polygon with vertices

$$\left(\sum_{i=1}^{j} r_i, \sum_{i=1}^{j} s_i\right)$$

for $j \in \{0, \ldots, n\}$.

Example 10.1.9. Let E be an elliptic curve over an algebraically closed field k of characteristic P. Let $K_0 = W(k)[1/p]$. Let $D = H^1_{\mathrm{cris}}(E/(W(k)) \otimes_{W(k)} K_0$. Then $D \cong D_{1,0} \oplus D_{1,1}$ if E is ordinary, and $D \cong D_{2,1}$ if E is supersingular.

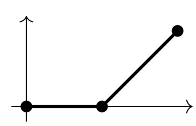


Definition 10.1.10. Let D be a finite-dimensional vector space equipped with a decreasing filtration that is separated and exhaustive. The *Hodge polygon* $P_H(D)$ of D is the polygon with endpoints

$$\left(\sum_{i<j} \dim \mathrm{gr}_i V, \sum_{i<j} i \dim \mathrm{gr}_i V\right)$$

for $j \in \mathbb{Z}$.

Example 10.1.11. Let E be an elliptic curve over a field K, and let $D = H^1_{\mathrm{dR}}(E)$. Then $\mathrm{gr}_0 D$ and $\mathrm{gr}_1 D$ are one-dimensional and all other graded pieces are zero.



Definition 10.1.12. A filtered ϕ-module D is *admissible* if for each subobject $D' \subseteq D$, $P_H(D')$ lies below $P_N(D')$, and the rightmost vertices of $P_H(D)$ and $P_N(D)$ coincide.

Remark 10.1.13. Although the category of filtered ϕ-modules is not abelian, the category of admissible filtered ϕ-modules is abelian.

Theorem 10.1.14. *The functor D_{cris} induces an equivalence of categories between the category of crystalline G_K-representations and the category of admissible filtered ϕ-modules.*

Example 10.1.15. We will classify two-dimensional admissible filtered ϕ-modules D over $\mathbb{Q}_p$ with Hodge–Tate weights 0 and 1. In order for the Newton and Hodge polygons to have the same endpoints, the p-adic valuations of the eigenvalues of ϕ must sum to 1. In order for the Newton polygon to lie above the Hodge polygon, the valuations must be either 0 and 1 or $\frac{1}{2}$ and $\frac{1}{2}$.

In the latter case, D has no subobjects, so any choice of one-dimensional subspace $\mathrm{Fil}^1 D$ makes D admissible. Any two choices yield isomorphic filtered ϕ-modules.

In the former case, the ϕ-eigenspaces are subobjects, and D is admissible if and only if $\mathrm{Fil}^1 D$ is not the eigenspace whose eigenvalue has valuation 0. If $\mathrm{Fil}^1 D$ is the other eigenspace, then D is a direct sum of two one-dimensional admissible filtered ϕ-modules. If $\mathrm{Fil}^1 D$ is not an eigenspace, then D is a nonsplit extension of two one-dimensional admissible filtered ϕ-modules. Any two choices of $\mathrm{Fil}^1 D$ that are not eigenspaces yield isomorphic filtered ϕ-modules.

11. B_{cris}

11.1. **The ring B_{cris}.** Let A^0_{cris} be the divided power envelope of $A_{\mathrm{inf}} = W(\mathcal{O}_{C^\flat})$ with respect to $\ker\Theta$, i.e. we adjoin $\frac{x^n}{n!}$ for all $x \in \ker\Theta$ and all positive integers n.

Define

$$A_{\mathrm{cris}} := \varprojlim_n A^0_{\mathrm{cris}}/p^n$$
$$B^+_{\mathrm{cris}} := A_{\mathrm{cris}}[1/p]$$

Proposition 11.1.1. *There are natural inclusions*

$$A^0_{\mathrm{cris}} \hookrightarrow A_{\mathrm{cris}} \hookrightarrow B^+_{\mathrm{cris}} \hookrightarrow B^+_{\mathrm{cris}} \otimes_{K_0} K \hookrightarrow B^+_{\mathrm{dR}}\,.$$

Proof. The ring A^0_{cris} is naturally a subring of $A_{\mathrm{inf}}[1/p]$, hence also of B^+_{dR}. The inclusion $A^0_{\mathrm{cris}} \hookrightarrow B^+_{\mathrm{dR}}$ is continuous for the p-adic topology on A^0_{cris} and the canonical topology on B^+_{dR}. Hence the inclusion factors as $A^0_{\mathrm{cris}} \hookrightarrow A_{\mathrm{cris}} \to B^+_{\mathrm{dR}}$.

Since p is not a zero divisor in A^0_{cris}, it is also not a zero divisor in A_{cris}, so A_{cris} injects into B^+_{cris}.

I don't know of a reference for the injectivity of $B^+_{\mathrm{cris}} \to B^+_{\mathrm{cris}} \otimes_{K_0} K \to B^+_{\mathrm{dR}}$. In [BC, Thm. 9.1.5] it is claimed that one can give a proof similar to that of [Fon82, §4.7] (which proves that the ring $B^+_{\max}$ defined below injects into B^+_{dR}). □

Corollary 11.1.2. $(B^+_{\mathrm{cris}})^{G_K} = (\operatorname{Frac} B^+_{\mathrm{cris}})^{G_K} = K_0$

Proof. Since there are G_K-equivariant injections $K_0 \hookrightarrow A_{\mathrm{inf}}[1/p] \hookrightarrow B^+_{\mathrm{cris}} \hookrightarrow B^+_{\mathrm{dR}}$, we see that $K_0 \subseteq (B^+_{\mathrm{cris}})^{G_K} \subseteq K$. Since $B^+_{\mathrm{cris}} \otimes_{K_0} K$ injects into B^+_{dR}, it must be the case that $(B^+_{\mathrm{cris}})^{G_K} = K_0$. By a similar argument, $(\operatorname{Frac} B^+_{\mathrm{cris}})^{G_K} = K_0$. □

Lemma 11.1.3. *A^0_{cris} is the ring obtained by adjoining $\frac{[\tilde p]^n}{n!}$ to A_{inf} for all integers n. Here $\tilde p = (p, p^{1/p}, p^{1/p^2}, \dots)$.*

Proof. Recall from the proof of Proposition 4.1.2 that $\ker\Theta$ is generated by $p - [\tilde p]$. Then the lemma follows after observing that $\frac{p^n}{n!} \in \mathbb{Z}_p$ for each n. □

Lemma 11.1.4. *Let $\phi\colon A_{\mathrm{inf}}[1/p] \to A_{\mathrm{inf}}[1/p]$ be the Frobenius map. Then $\phi(A^0_{\mathrm{cris}}) \subseteq A^0_{\mathrm{cris}}$.*

Proof. Apply Lemma 11.1.3. It is clear that $\frac{[\tilde p]^{pn}}{n!} \in A^0_{\mathrm{cris}}$ for each n. □

In particular, there is a Frobenius action on A_{cris} and B^+_{cris}.

Lemma 11.1.5. *For any $x \in A_{\mathrm{cris}} \cap \ker \Theta_{\mathbb{Q}}$, the power series*

$$\log(1+x) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}x^n}{n}$$

converges in A_{cris}.

Proof. This follows from the fact that $\frac{n!}{n} = (n-1)!$ goes to zero p-adically as $n \to \infty$. □

In particular, $t = \log[\epsilon] \in A_{\mathrm{cris}}$. We then define

$$B_{\mathrm{cris}} := B_{\mathrm{cris}}^{+}[1/t]\,.$$

Remark 11.1.6. In fact, t^{p-1} is a multiple of p in A_{cris} [BC, Prop. 9.1.3] [FO, Prop. 6.6], so $B_{\mathrm{cris}} = A_{\mathrm{cris}}[1/t]$.

Lemma 11.1.7. *The Frobenius map $\phi\colon A_{\mathrm{cris}} \to A_{\mathrm{cris}}$ is injective.*

Proof. A proof is unfortunately missing from [BC], but see the proof of Lemma 11.1.9 below. □

The ring B_{cris} is not particularly nice. Sometimes it is easier to work with the ring $B_{\max}$, defined as follows.

$$B_{\max}^{+} := \left\{ \sum_{n=0}^{\infty} x_n \frac{\xi^n}{p^n} \in B_{\mathrm{dR}}^{+},\, x_n \in A_{\mathrm{inf}}[1/p],\, \lim_{n\to\infty} x_n = 0 \right\}$$

Here, "$x_n \to 0$" means that all but finitely any of the x_n are in A_{inf} and the sequence goes to zero for the $(p, [\tilde{p}])$-adic topology on A_{inf}.

$$B_{\max} := B_{\max}^{+}[1/t]$$

Remark 11.1.8. $B_{\max}^{+}$ is a Huber ring.

Lemma 11.1.9.

(1) $\varphi(B_{\max}^{+}) \subset B_{\mathrm{cris}}^{+} \subset B_{\max}^{+}$.

(2) *For any $\mathbb{Q}_p$-representation of G_K, $(V \otimes B_{\mathrm{cris}})^{G_K} \to (V \otimes B_{\max})^{G_K}$ is an isomorphism.*

Proof. Item (1) follows from the inequality

$$\frac{n}{p-1} - \log_p(n+1) \le v_p(n!) \le \frac{n}{p-1}\,.$$

(See for example [FF18, Prop. 1.10.12].) Since the Frobenius operator on $B_{\max}^{+}$ is injective [FF18, §1.10.1], the Frobenius operator on $(V \otimes B_{\max})^{G_K}$ is also injective. Since this vector space is finite-dimensional, the Frobenius must be an isomorphism. Then item (2) follows from item (1). □

Define a filtration on B_{cris} by pulling back the filtration from B_{dR}.

Remark 11.1.10. $\mathrm{Fil}^0 B_{\mathrm{cris}}$ is strictly larger than B_{cris}^{+}; see [Dia17, Problem 25].

Proposition 11.1.11. *The sequence*

$$0 \to \mathbb{Q}_p \to \mathrm{Fil}^0 B_{\mathrm{cris}} \xrightarrow{\varphi - 1} B_{\mathrm{cris}} \to 0$$

is exact.

Proof. See [FO, Thm. 6.25]. □

Corollary 11.1.12. *D_{cris} induces a fully faithful functor from the category of crystalline representations of G_K to the category of filtered ϕ-modules.*

We already mentioned the stronger statement Theorem 10.1.14, but that is substantially harder to prove.

Proof. If V is crystalline, then the natural map $B_{\mathrm{cris}} \otimes_{K_0} D_{\mathrm{cris}}(V) \to B_{\mathrm{cris}} \otimes_{\mathbb{Q}_p} V$ is a Galois- and Frobenius-equivariant isomorphism of filtered vector spaces. Taking the ϕ-invariant subspace of Fil^0 gives $\mathrm{Fil}^0(B_{\mathrm{cris}} \otimes_{K_0} D_{\mathrm{cris}}(V))^{\phi=1} \cong V$. This gives us a left inverse of D_{cris}. □

One can also use Proposition 11.1.11 to prove that if V is crystalline, then $D_{\mathrm{cris}}(V)$ is admissible. Again, the converse is much harder to prove.

Example 11.1.13. In example 10.1.4, we showed that unramified representations of G_K are crystalline. Moreover, since the image of $D_{\mathcal{E}}$ consists of étale ϕ-modules, an unramified representation must have all Hodge–Tate weights equal to zero. Conversely, the full faithfulness of D_{cris} implies that any crystalline representation with all Hodge–Tate weights equal to zero is unramified.

Example 11.1.14. It is not difficult to construct representations that are potentially unramified (i.e. the restriction to some open subgroup of G_K is unramified) but not unramified. Such representations are potentially crystalline but not crystalline. They are also de Rham since any potentially de Rham representation is de Rham by Lemma 5.2.9.

11.2. **B_{st}.** So far, we have defined a ring B_{dR} that contains the periods of all algebraic varieties over K, and a ring B_{cris} that contains the periods of all proper smooth varieties with good reduction over K. We will now describe an intermediate ring B_{st} that contains the periods of all proper smooth varieties with semistable reduction over K.

There is a G_K- and Frobenius-equivariant group homomorphism $\log\colon (\mathcal{O}_C^\flat)^\times \to B_{\mathrm{cris}}$, defined as follows. If $x \in \mathcal{O}_C^\flat$ satisfies $|x-1| < 1$, then the power series for $\log[x]$ converges in B_{cris}, and we define $\log x := \log[x]$. We extend the logarithm to $(\mathcal{O}_C^\flat)^\times$ by setting $\log x = 0$ for x in the image of the residue field of C.

If we want to extend the logarithm to $(C^\flat)^\times$, we need to replace B_{cris} with a larger ring.

Definition 11.2.1. The ring B_{st} is the polynomial ring $B_{\mathrm{cris}}[\lambda]$, with Galois action given by

$$g \cdot \lambda = \lambda + \log\left[\frac{g \cdot \tilde{p}}{\tilde{p}}\right]$$

and Frobenius action given by $\phi(\lambda) = p\lambda$.

One can then define a G_K- and Frobenius-equivariant logarithm map $\log\colon (C^\flat)^\times \to B_{\mathrm{st}}$ sending $\tilde{p} \mapsto \lambda$.

12. Semistable representations, adic spaces

12.1. **Properties of B_{st} and semistable represntations.** Next, we embed $B_{\mathrm{st}} \otimes_{K_0} K$ in B_{dR}. The power series for $\log[\tilde{p}]$ does not converge in B_{dR}. However, the power series for $\log([\tilde{p}]/p)$ does converge. For any $x \in K$, the map $B_{\mathrm{st}} \otimes_{K_0} K \to B_{\mathrm{dR}}$ that extends the usual map $B_{\mathrm{cris}} \otimes_{K_0} K \to B_{\mathrm{dR}}$ and sends $\lambda \to \log([\tilde{p}]/p) + x$ is

G_K-equivariant. We (somewhat arbitrarily) choose $x = 0$. (See [FO, Lem. 6.12] for a proof that this map is injective.)

The different embeddings of $B_{\mathrm{st}} \otimes_{K_0} K$ into B_{dR} are related by an action of K on $B_{\mathrm{st}} \otimes_{K_0} K$. Under this action, $x \in K$ sends $\lambda \mapsto \lambda + x$. We define the *monodromy operator* $N\colon B_{\mathrm{st}} \to B_{\mathrm{st}}$ to be the unit tangent vector of this group action, i.e N is the derivation that annihilates B_{cris} and sends $\lambda \mapsto 1$. The operator N satisfies $N\phi = p\phi N$.

One reason for studying B_{st} is the following theorem.

Theorem 12.1.1. *Any de Rham representation V of G_K is potentially semistable, i.e. there exists a finite extension L of G_K so that $\dim_{L_0}(V \otimes_{\mathbb{Q}_p} B_{\mathrm{st}})^{G_L} = \dim_{\mathbb{Q}_p} V$. Here $L_0 = (B_{\mathrm{st}})^{G_L} = W(\ell)[1/p]$, where ℓ is the residue field of L.*

Definition 12.1.2. A (ϕ, N)-*module* over K_0 is an isocrystal D over K_0 equipped with at K_0-linear endomorphism $N\colon D \to D$ satisfying $N\phi = p\phi N$.

A *filtered* (ϕ, N)-*module* over K is a pair consisting of a (ϕ, N)-module D over K_0 and a filtration on $D \otimes_{K_0} K$.

Definition 12.1.3. Let V be a $\mathbb{Q}_p$-representation of G_K. Then $D_{\mathrm{st}}(V)$ is the filtered (ϕ, N)-module with underlying (ϕ, N)-module $(V \otimes_{\mathbb{Q}_p} B_{\mathrm{st}})^{G_K}$, with the filtration inherited from $D_{\mathrm{dR}}(V)$.

Remark 12.1.4. Different embeddings $B_{\mathrm{st}} \otimes_{K_0} K \hookrightarrow B_{\mathrm{dR}}$ induce different filtrations on $B_{\mathrm{st}} \otimes_{K_0} K$, but these are all related by the action of the additive group K on $B_{\mathrm{st}} \otimes_{K_0} K$. Since $D_{\mathrm{st}}(V)$ is a finite-dimensional K_0-vector space and ϕ is injective, the identity $N\phi = p\phi N$ implies that the action of N on $D_{\mathrm{st}}(V)$ must be nilpotent. Then we can recover the action of K on $D_{\mathrm{st}}(V) \otimes_{K_0} K$ by exponentiating N. So any two choices of embedding $B_{\mathrm{st}} \otimes_{K_0} K \hookrightarrow B_{\mathrm{dR}}$ yield naturally isomorphic D_{st} functors.

Definition 12.1.5. A $\mathbb{Q}_p$-representation V of G_K is *semistable* if $\dim_{K_0}(V \otimes_{\mathbb{Q}_p} B_{\mathrm{st}})^{G_K} = \dim_{\mathbb{Q}_p} V$.

Definition 12.1.6. A filtered (ϕ, N)-module D is *admissible* if for each subobject $D' \subseteq D$, $P_H(D')$ lies below $P_N(D')$, and the rightmost vertices of $P_H(D)$ and $P_N(D)$ coincide.

Theorem 12.1.7. *A filtered (ϕ, N)-module D is admissible if and only if it is isomorphic to $D_{\mathrm{st}}(V)$ for some semistable representation V of G_K.*

Example 12.1.8. Choose $q \in K$ with $|q| < 1$. One can take the quotient $\mathbb{G}_m/q^{\mathbb{Z}}$ in the category of rigid analytic spaces, and the result is an elliptic curve E. Then the Tate module $T(E)$ is isomorphic to the p-adic completion of the subgroup of $(C^\flat)^\times$ generated by ϵ and $\tilde{q} = (q, q^{1/p}, \dots)$. Then $T(E)$ is isomorphic as a G_K-representation to the $\mathbb{Z}_p$-submodule of B_{st} generated by $\log \epsilon$ and $\log \tilde{q}$. In particular, $D_{\mathrm{st}}(H^1_{\text{ét}}(E_{\overline{K}}, \mathbb{Q}_p)) \cong \mathrm{Hom}_{G_K}(T(E), B_{\mathrm{st}})$ is two-dimensional, generated by log and $N \circ \log$. Hence $H^1_{\text{ét}}(E_{\overline{K}}, \mathbb{Q}_p)$ is semistable.

Note that the (ϕ, N)-module structure of $D_{\mathrm{st}}(H^1_{\text{ét}}(E_{\overline{K}}, \mathbb{Q}_p))$ does not depend on q, but the filtration does depend on q.

12.2. **Adic spaces.** In the remaining part of the course, we will do some geometry. We are intersted in the following spaces:

- Rigid analytic spaces, which are a p-adic analogue of complex manifolds.

- Very large covers of rigid analytic spaces, especially perfectoid spaces.
- The Fargues–Fontaine curve, which parameterizes untilts of a perfectoid field K of characteristic p.

These all fit into the framework of adic spaces, which we will define next.

Definition 12.2.1. A *Huber ring* is a topological ring A such that there exists an open subring $A_0 \subset A$ and a finitely generated ideal $I \subset A_0$ so that A_0 has the I-adic topology (powers of I form a basis of neighborhoods of the identity).

We say that A_0 is a *ring of definition* of A and I is an *ideal of definition* of A_0.

Example 12.2.2. The following topological rings are Huber rings.

- Any ring with the discrete topology ($A_0 =$ any subring, $I = 0$).
- $\mathbb{Z}_p$, ($A_0 = \mathbb{Z}_p$, $I = (p)$)
- $\mathbb{Q}_p$, ($A_0 = \mathbb{Z}_p$, $I = (p)$)
- $\mathbb{Q}_p \langle T \rangle$, the ring of analytic functions converging on the closed unit disc ($A_0 = \mathbb{Z}_p \langle T \rangle$, $I = (p)$)
- $W(\mathcal{O}_{K^\flat})$ for a perfectoid field K ($A = W(\mathcal{O}_{K^\flat})$, $I = (p, [\pi])$ where $\pi \in \mathcal{O}_{K^\flat}$ satisfies $0 < |\pi| < 1$)

Definition 12.2.3. Let A be a ring. A *valuation* on A is a map $|\cdot|\colon A \to \Gamma \cup \{0\}$, where Γ is a totally ordered abelian group (written multiplicatively), satisfying the following properties:

(1) $|xy| = |x||y|$ for all $x, y \in A$
(2) $|x + y| \leq \max(|x|, |y|)$ for all $x, y \in A$
(3) $|0| = 0$, $|1| = 1$

If A is a topological ring, then we say that a valuation is *continuous* if for all $\gamma \in \Gamma$, $\{a \in A : |a| < \gamma\}$ is open.

We say that two valuations $|\cdot|$ and $|\cdot|'$ are *equivalent* if $|a| \leq |b| \iff |a|' \leq |b|'$ for all $a, b \in A$.

Definition 12.2.4. Let A be a topological ring. Define $\mathrm{Cont}(A)$ to be the set of equivalence classes of continuous valuations of A.

Give $\mathrm{Cont}(A)$ the topology with a sub-basis of open sets consisting of sets the form

$$\{x | |f(x)| \leq |g(x)| \neq 0\} \ .$$

for $f, g \in A$. Here $|f(x)|$, $|g(x)|$ denote the valuations of f and g under a representative of the equivalence class x.

Definition 12.2.5. Let A be a topological ring. A subset S of A is *bounded* if for all open neighborhoods U of 0, there is an open neighborhood V of 0 such that $VS \subset U$.

Definition 12.2.6. Let A be a Huber ring. An element $f \in A$ is *power-bounded* if $\{f^n | n \in \mathbb{N}\}$ is bounded.

We will write A° for the subring of power-bounded elements of A.

Definition 12.2.7. Let A be a Huber ring. A subring $A^+ \subset A^\circ$ is a *ring of integral elements* if it is open and integrally closed in A.

A *Huber pair* is a pair (A, A^+), where A is a Huber ring, and $A^+ \subset A$ is a ring of integral elements.

For a Huber pair (A, A^+), define $\mathrm{Spa}(A, A^+) \subset \mathrm{Cont}(A)$ to be the subspace consisting of those valuations x for which $|f(x)| \leq 1$ for all $f \in A^+$.

13. Adic spaces, Fargues–Fontaine curve

13.1. **Adic spaces.**

Definition 13.1.1. A *rational subset* of $\mathrm{Spa}(A, A^+)$ is a subset defined by inequalities $|a_1|, |a_2|, \ldots, |a_n| \le |a_0| \neq 0$, where $a_0, a_1, \ldots, a_n \in A$ generate an open ideal.

Lemma 13.1.2. *Rational subsets form a basis for the topology of* $\mathrm{Spa}(A, A^+)$.

Proposition 13.1.3. *Let* $X = \mathrm{Spa}(A, A^+)$*, and let* U *be a rational subset of* X*. There exists a complete Huber pair* (B, B^+) *over* (A, A^+) *such that* $\mathrm{Spa}(B, B^+) \to \mathrm{Spa}(A, A^+)$ *factors through* U*, and is universal for such maps. Moreover, the map* $\mathrm{Spa}(B, B^+) \to U$ *is a homeomorphism.*

Definition 13.1.4. For any Huber pair (A, A^+), we define a presheaf of topological rings $\mathcal{O}_X$ on $X = \mathrm{Spa}(A, A^+)$ as follows. If U is rational, we define $\mathcal{O}_X(U)$ to be the ring B of Proposition 13.1.3. For general U, we define

$$\mathcal{O}_X(U) = \varprojlim_{W \subset U \text{ rational}} \mathcal{O}_X(W)\,.$$

We define $\mathcal{O}_X^+(U)$ similarly.

This presheaf $\mathcal{O}_X$ is not a sheaf in general. But in all examples that we will consider in this course, $\mathcal{O}_X$ will be a sheaf. If $\mathcal{O}_X$ is a sheaf, then $\mathcal{O}_X^+$ is also a sheaf.

Lemma 13.1.5. *The completion* $(A, A^+) \to (\hat{A}, \hat{A}^+)$ *induces a canonical isomorphism* $\mathrm{Spa}(\hat{A}, \hat{A}^+) \to \mathrm{Spa}(A, A^+)$.

Let $X = \mathrm{Spa}(A, A^+)$*. Then* $\mathcal{O}_X(X) \cong \hat{A}$*,* $\mathcal{O}_X^+(X) \cong \hat{A}^+$*.*

Definition 13.1.6. A *v-ringed space* is a triple $(X, \mathcal{O}_X, (|\cdot(x)|)_{x\in X})$ where X is a topological space, $\mathcal{O}_X$ is a sheaf of topological rings on X, and for each $x \in X$, $|\cdot(x)|$ is an equivalence class of continuous valuations on $\mathcal{O}_X(x)$. A morphism of v-ringed spaces is a a morphism of ringed spaces, such that the maps on sections are continuous, and valuations are preserved.

Definition 13.1.7. An *affinoid adic space* is a v-ringed space that is isomorphic to some $\mathrm{Spa}(A, A^+)$. (In particular, we require that the structure presheaf is a sheaf.)

An *adic space* is a v-ringed space that has an open covering by affinoid adic spaces.

13.2. **Rigid analytic spaces.**

Let K be a nonarchimedan field. The closed unit disc over K is defined to be the adic space $D := \mathrm{Spa}(K\langle T\rangle, \mathcal{O}_K\langle T\rangle)$, where

$$K\langle T\rangle = \left\{ \sum_{n=0}^{\infty} a_n T^n \,\middle|\, a_n \in K, \lim_{n\to\infty} a_n = 0 \right\}$$

and $\mathcal{O}_K\langle T\rangle = K\langle T\rangle^\circ$ is the subring consisting of those series with all $a_n \in \mathcal{O}_K$.

Let $C = \widehat{\overline{K}}$. For any $z \in \mathcal{O}_C$, $f \mapsto |f(z)|$ determines a point of D (and two elements that are related by the Galois action determine the same point). These are not the only points of D. For example, there is the Gauss point η corresponding to the norm

$$\sum a_n T^n \mapsto \sup_n |a_n| = \sup_{z\in\mathcal{O}_C} \left| \sum_n a_n z^n \right| .$$

More generally, for any closed disc $D' \subseteq D(C)$, $f \mapsto \sup_{z \in D'} |f(z)|$ also determines a point of D. The underlying topological space of D is a sort of infinite tree. In partcular, it is connected, unlike $\mathcal{O}_C$.

Example 13.2.1. Let $C = \widehat{\overline{K}}$ for some p-adic field K. The points on the closed unit disc over C can be classified into five types:

1. For any $\alpha \in \mathcal{O}_C$, $f \mapsto |f(\alpha)|$.
2. The supremum norm on a disc with radius in $|C^\times|$.
3. The supremum norm on a disc with radius not in $|C^\times|$.
4. The limit of supremum norms on a decreasing sequence of discs with empty intersection ("dead ends").
5. Tangent directions to type 2 points: for $\alpha \in \mathcal{O}_C$, $r \in |C^\times|$ with $r \leq 1$:
$$\sum a_n (T-\alpha)^n \mapsto \max_n |a_n| r^n \gamma^n$$
where γ is either infinitesimally larger or smaller than 1.

All points except type 2 points are closed. The closure of a type 2 point consists of type 5 points, and looks like $\mathbb{P}^1_{\overline{\mathbb{F}}_p}$ (or $\mathbb{A}^1_{\overline{\mathbb{F}}_p}$ for the Gauss point).

Definition 13.2.2. A topological K-algebra is *topologically of finite type* if it is of the form $K\langle T_1, \dots, T_n \rangle / I$ for some n and some ideal $I \subset K\langle T_1, \dots, T_n\rangle$.

Definition 13.2.3. Let K be a nonarchimedean field. A *rigid analytic space* over K is an adic space X over (K, K°) such that X has a covering by open sets of the form $\mathrm{Spa}(A, A^\circ)$, where A is topologically of finite type over K.

Proposition 13.2.4. *Let K be a nonarchimedean field. There is a faithful functor $(\cdot)^{\mathrm{ad}}$ from the category of varieties over K to the category of rigid analytic spaces over (K, K°).*

Example 13.2.5.

1. $(\mathbb{P}^1_k)^{\mathrm{ad}}$ is formed by gluing two closed discs along an annulus.
2. $(\mathbb{A}^1_k)^{\mathrm{ad}}$ is an increasing union of closed discs.

13.3. **The Fargues–Fontaine curve.**

Definition 13.3.1. Let K be a perfectoid field of characteristic p.
$$\begin{aligned} \mathcal{Y}_{[0,\infty]} &:= \mathrm{Spa}(W(\mathcal{O}_K), W(\mathcal{O}_K)) \setminus \{|p| = |\pi| = 0\} \\ \mathcal{Y}_{(0,\infty)} &:= \mathcal{Y}_{[0,\infty]} \setminus \{|p[\pi]| = 0\} \\ \mathcal{X} &:= \mathcal{Y}_{(0,\infty)} / \phi^{\mathbb{Z}}. \end{aligned}$$
Here π is any element of $\mathcal{O}_K$ satisfying $0 < |\pi|_K < 1$. We call $\mathcal{X}$ the *adic Fargues-Fontaine curve* for K.

More generally, for an interval $I \subset \mathbb{R}_{\geq 0} \cup \{\infty\}$, we define $\mathcal{Y}_I$ to be the the (open) subspace of $\mathcal{Y}$ satisfying $|\log[\pi]|/|\log p| \in I$.

Remark 13.3.2. The group $\phi^{\mathbb{Z}}$ acts properly discontinuously on $\mathcal{Y}_{(0,\infty)}$, so there are no issues with taking the quotient.

Remark 13.3.3. One can also consider an equal-characteristic Fargues–Fontaine curve, in which $W(\mathcal{O}_K)$ is replaced with $\mathcal{O}_K [\![T]\!]$.

Example 13.3.4. Let $K^\sharp$ be an untilt of K. Then there is a point $x_{K^\sharp}$ corresponding to the valuation $z \mapsto |\Theta(z)|$. In the case $K^\sharp = C$, $K = C^\flat$, we have $\widehat{\mathcal{O}}_{\mathcal{X}, x_C} \cong B^+_{\mathrm{dR}}$.

Example 13.3.5. Let $K = C^\flat$, and let $\pi = \tilde{p}$, so that $\mathcal{Y}_{[1,\infty]} \subset \mathcal{Y}_{[0,\infty]}$ is open subspace defined by the inequality $|[\tilde{p}]| \leq |p| \neq 0$. Then $\mathcal{O}_{\mathcal{Y}_{[0,\infty]}}(\mathcal{Y}_{[1,\infty]}) \cong B^+_{\max}$.

14. Fargues–Fontaine curve continued

14.1. **Vector bundles on the Fargues–Fontaine curve.** Define a functor $\mathcal{E}$ from the category of isocrystals over $W(K)[1/p]$ to the category of vector bundles on $\mathcal{X}$ as follows. Given an isocrystal D over $W(K)[1/p]$, $D \otimes_{W(K)[1/p]} \mathcal{O}_{\mathcal{Y}_{(0,\infty)}}$ is a ϕ-equivariant vector bundle on $\mathcal{Y}_{(0,\infty)}$. This bundle descends to a vector bundle $\mathcal{E}(D)$ on X.

Theorem 14.1.1.

(1) *$\mathcal{E}$ is faithful (but not full).*
(2) *If K is algebraically closed, then $\mathcal{E}$ induces a bijection between isomorphism classes of isocrystals over $W(K)[1/p]$ and vector bundles on $\mathcal{X}$; hence there is a Dieudonné–Manin decomposition for vector bundles on $\mathcal{X}$.*

We will write $\mathcal{O}(-s/r)$ for $\mathcal{E}(D_{r,s})$.

Note the resemblance to the classification of vector bundles on $\mathbb{P}^1$.

Theorem 14.1.2 (Grothendieck). *Let k be an algebraically closed field. Every vector bundle on $\mathbb{P}^1_k$ is a direct sum of line bundles of the form $\mathcal{O}(n)$ for some $n \in \mathbb{Z}$.*

Proposition 14.1.3.

- $H^0(\mathcal{X}, \mathcal{O}(0)) \cong \mathbb{Q}_p$
- $H^0(\mathcal{X}, \mathcal{O}(\lambda)) = 0$ *if* $\lambda < 0$.
- $H^0(\mathcal{X}, \mathcal{O}(\lambda))$ *is an infinite-dimensional* $\mathbb{Q}_p$*-vector space if* $\lambda > 0$.
- $H^1(\mathcal{X}, \mathcal{O}(\lambda)) = 0$ *if* $\lambda \geq 0$.
- $H^1(\mathcal{X}, \mathcal{O}(\lambda))$ *is an infinite-dimensional* $\mathbb{Q}_p$*-vector space if* $\lambda < 0$.
- $H^i(\mathcal{X}, \mathcal{O}(\lambda)) = 0$ *for* $i > 1$.

Remark 14.1.4. These cohomology groups are examples of *Banach–Colmez* spaces. Roughly, for any untilt $K^\sharp$, these are extenions or quotients of $K^\sharp$-vector spaces by $\mathbb{Q}_p$-vector spaces. For example, assuming $\mathbb{Q}_p^{\mathrm{cyc}} \subseteq K^\sharp$, there are exact sequences

$$0 \to \mathbb{Q}_p(1) \to H^0(\mathcal{X}, \mathcal{O}(1)) \to K^\sharp \to 0$$

$$0 \to H^1(\mathcal{X}, \mathcal{O}) \to K^\sharp(-1) \to \mathbb{Q}_p(-1) \to 0$$

The exact sequences depend on the choice of untilt: different untilts give different choices of one-dimensional $\mathbb{Q}_p$-vector subspaces of $H^0(\mathcal{X}, \mathcal{O}(1))$.

Corollary 14.1.5. *Assume K is algebraically closed, and let G be a closed subgroup of* $\operatorname{Aut} K$*. Then there is an equivalence of categories between $\mathbb{Q}_p$-representations of G and G-equivariant vector bundles on $\mathcal{X}$ that are trivial as vector bundles.*

Now let K be a p-adic field, and let $C = \widehat{\overline{K}}$. As mentioned in Example 13.3.4, there is a point x_C on $\mathcal{X}_{C^\flat}$ corresponding to $z \mapsto |\Theta(z)|$.

Definition 14.1.6. A *modification of vector bundles* on $\mathcal{X}$ at x_C is a meromorphic map of vector bundles $\mathcal{E} \dashrightarrow \mathcal{F}$ that is an isomorphism away from x_C.

Example 14.1.7. The power series for $t = \log[\epsilon]$ converges on $\mathcal{Y}_{(0,\infty)}$. Then multiplication by t induces a modification of vector bundles $\mathcal{O} \to \mathcal{O}(1)$ on X.

Example 14.1.8.

(1) Given a de Rham representation V of G_K, we can define a G_K-equivariant modification of vector bundles on $\mathcal{X}$ at x_C as follows. Let $\mathcal{F} := \mathcal{O} \otimes_{\mathbb{Q}_p} V$. To define a modification of vector bundles $\mathcal{E} \dashrightarrow \mathcal{F}$ at x_C, we just need to define a B_{dR}^+-lattice inside $V \otimes_{\mathbb{Q}_p} B_{\mathrm{dR}}$. (The trivial modification would correspond to $V \otimes_{\mathbb{Q}_p} B_{\mathrm{dR}}^+$.) We choose the lattice $D_{\mathrm{dR}}(V) \otimes_K B_{\mathrm{dR}}^+$.

(2) The representation V is potentially semistable, so we can find a finite Galois extension L/K so that the restriction of V to G_L is semistable. Let $D_{\mathrm{pst}}(V) = (V \otimes_{\mathbb{Q}_p} B_{\mathrm{st}})^{G_L}$; it is a filtered (ϕ, N)-module over L with an action of $\mathrm{Gal}(L/K)$.

(3) Given a filtered (ϕ, N)-module D over L with an action of $\mathrm{Gal}(L/K)$, we can define a modification of vector bundles as follows. Observe that $(D \otimes_{L_0} B_{\mathrm{st}}^+)^{N=0}$ is a finite free B_{cris}^+-module with semilinar G_K and Frobenius actions. Since $B_{\mathrm{cris}}^+ \subset B_{\max}^+$, this module determines a vector bundle on $\mathcal{Y}_{[1,\infty]}$. After pulling back along the Frobenius, we get a G_K- and Frobenius-equivariant vector bundle on $\mathcal{Y}_{(0,\infty]}$, which descends to a vector bundle $\mathcal{E}$ on $\mathcal{X}$. Specifying a modification $\mathcal{E} \to \mathcal{F}$ is equivalent to specifying a B_{dR}^+ lattice inside $D \otimes_{L_0} B_{\mathrm{dR}}$. We choose the one induced by the filtration on $D \otimes_{L_0} L$.

Theorem 14.1.9. *Let V be a de Rham representation of G_K. Then the modifications of vector bundles associated with V under Example 14.1.8(1) and with $D_{\mathrm{pst}}(V)$ under Example 14.1.8(3) are naturally isomorphic.*

In the situation of Example 14.1.8(3), the slopes of $\mathcal{E}$ are minus the slopes of D. The bundle $\mathcal{F}$ is pure of slope zero if and only if D is admissible.

15. More on the Fargues–Fontaine curve, prismatic cohomology

15.1. Schematic Fargues–Fontaine curve.

Definition 15.1.1. Let K be a perfectoid field. The *schematic Fargues-Fontaine curve* is defined by

$$X = \mathrm{Proj} \bigoplus_{n=0}^{\infty} H^0(\mathcal{X}, \mathcal{O}_{\mathcal{X}}(n))$$

Proposition 15.1.2.

(1) *X is a Noetherian scheme of Krull dimension one.*

(2) *Let $x \in X$, and let k_x be its residue field. Then k_x is perfectoid and $k_x^\flat$ is a finite extension of K. (In particular, X is not locally of finite type over $\mathbb{Q}_p$).*

(3) *If K is algebraically closed, then the closed points of X are in bijection with isomorphism classes of untilts of K modulo Frobenius.*

(4) *X is complete in the sense that for any rational function f on X, $\sum_{x \in X} (\deg x)(\mathrm{ord}_x f) = 0$. Here $\mathrm{ord}_x f$ is the valuation of f in the DVR $\mathcal{O}_{X,x}$ and $\deg x = [k_x : K]$.*

(5) *There is an equivalence of categories between vector bundles on X and vector bundles on $\mathcal{X}$.*

15.2. Analogy with complex Hodge theory.

When thinking about the Fargues–Fontaine curve, it can sometimes be helpful to think about its archimedean analogue, the twistor line.

Let $\tilde{\mathbb{P}}^1_{\mathbb{R}} := \mathrm{Proj}\, \mathbb{R}[X, Y, Z]/(X^2 + Y^2 + Z^2)$. It is the unique (up to isomorphism) genus zero curve over $\mathbb{R}$ with no real points. We can also think of it as the quotient of $\mathbb{P}^1_{\mathbb{C}}$ by the antiholomorphic map $z \mapsto \bar{z}^{-1}$.

Choose an arbitrary point $\infty \in \tilde{\mathbb{P}}^1_{\mathbb{R}}$. For example, we can take ∞ to be the point $X = 0$. The group PGO_3 acts on $\tilde{\mathbb{P}}^1_{\mathbb{R}}$, and the stabilizer of ∞ is PGO_2. The group PGO_2 has two connected components. The identity component is PSO_2. We can identify $\mathrm{PGO}_2(\mathbb{C})$ with the Weil group of $\mathbb{R}$ and $\mathrm{PSO}_2(\mathbb{C})$ with the Weil group of $\mathbb{C}$.

Proposition 15.2.1. *The category of* PSO_2*-equivariant semistable vector bundles on* $\tilde{\mathbb{P}}^1_{\mathbb{R}}$ *is equivalent to the category of pure* $\mathbb{R}$*-Hodge structures.*

15.3. **Diamonds and the Fargues–Fontaine curve.**

Definition 15.3.1. Let X be an adic space. A point $x \in X$ is *analytic* if the kernel of the valuation $|\cdot|_x$ is not open. We say that X is *analytic* if every point of X is analytic.

Example 15.3.2. $\mathrm{Spa}(\mathbb{Z}_p, \mathbb{Z}_p)$ consists of two points. The generic point, corresponding to the p-adic valuation, is analytic. The closed point, corresponding to the discrete norm on $\mathbb{F}_p$, is not analytic. $\mathrm{Spa}(\mathbb{Q}_p, \mathbb{Z}_p)$ contains just the generic point, so it is analytic.

The closed disc $\mathrm{Spa}(\mathbb{Q}_p \langle T \rangle, \mathbb{Z}_p \langle T \rangle)$ is analytic. More generally, rigid analytic spaces are analytic.

Let K be a perfectoid field of characteristic p. $\mathrm{Spa}(W(\mathcal{O}_K), W(\mathcal{O}_K))$ is not analytic, but $\mathcal{Y}_{[0,\infty]}$ is analytic.

Definition 15.3.3. A *Tate ring* is a Huber ring containing a topologically nilpotent unit.

Proposition 15.3.4. *An adic space is analytic if and only if it can be covered by affinoids of the form* $\mathrm{Spa}(A, A^+)$ *with* A *Tate.*

Definition 15.3.5. A Huber ring is *uniform* if A° is bounded (equivalently, A° is a ring of definition).

Definition 15.3.6. A complete Tate $\mathbb{Z}_p$-algebra A is *perfectoid* if it is uniform, there exists a topologically nilpotent unit $\pi \in A^\times$ so that $\pi^p \mid p$ in A°, and the Frobenius map $A^\circ/\pi \to A^\circ/\pi^p$ is an isomorphism.

Definition 15.3.7. A *perfectoid space* is an adic space that can be covered by affinoids of the form $\mathrm{Spa}(A, A^+)$ with A perfectoid.

As with perfectoid fields, one can define the tilt of a perfectoid space.

Definition 15.3.8. Let Perf denote the category of characteristic p perfectoid spaces. Let X be an adic space over $\mathrm{Spa}(\mathbb{Z}_p, \mathbb{Z}_p)$. Define $X^\diamondsuit$ to be the functor $\mathrm{Perf} \to \mathrm{Set}$ sending $Y \in \mathrm{Perf}$ to the set of pairs consisting of an untilt $Y^\sharp$ of Y and a map $Y^\sharp \to X$.

For a Huber pair (A, A^+), we will write $\mathrm{Spd}(A, A^+)$ for $\mathrm{Spa}(A, A^+)^\diamondsuit$.

Remark 15.3.9. If X is analytic, then the functor $X^\diamondsuit$ is a "diamond". In particular, it is a sheaf on certain sites, including the "pro-étale site" and the "v-site".

Proposition 15.3.10.

$$\begin{aligned}
\mathrm{Spd}(W(\mathcal{O}_K), W(\mathcal{O}_K)) &\cong \mathrm{Spd}(\mathcal{O}_K, \mathcal{O}_K) \times \mathrm{Spd}(\mathbb{Z}_p, \mathbb{Z}_p) \\
\mathcal{Y}^{\lozenge}_{(0,\infty]} &\cong \mathrm{Spd}(\mathcal{O}_K, \mathcal{O}_K) \times \mathrm{Spd}(\mathbb{Q}_p, \mathbb{Z}_p) \\
\mathcal{Y}^{\lozenge}_{[0,\infty)} &\cong \mathrm{Spd}(K, \mathcal{O}_K) \times \mathrm{Spd}(\mathbb{Z}_p, \mathbb{Z}_p) \\
\mathcal{Y}^{\lozenge}_{(0,\infty)} &\cong \mathrm{Spd}(K, \mathcal{O}_K) \times \mathrm{Spd}(\mathbb{Q}_p, \mathbb{Z}_p)
\end{aligned}$$

In particular, there is a projection $\mathcal{Y}^{\lozenge}_{(0,\infty)} \to \mathrm{Spd}(K, \mathcal{O}_K)$, even though there is no map $\mathcal{Y}_{(0,\infty)} \to \mathrm{Spa}(K, \mathcal{O}_K)$.

Question 15.3.11. Let Z be a rigid analytic space over a nonarchimedean field K. Let $C = \widehat{\overline{K}}$. For any nonnegative integer i, the G_K-representation $H^i_{\text{ét}}(X_C, \mathbb{Q}_p)$ determines a ϕ-equivariant modification of vector bundles $\mathcal{E} \dashrightarrow \mathcal{F}$ on $\mathcal{Y}_{C^\flat,(0,\infty)}$. Can this modification be described as some sort of derived pushforward along the map

$$(Z_C)^{\lozenge} \times \mathrm{Spd}(\mathbb{Q}_p, \mathbb{Z}_p) \to \mathrm{Spd}(C, \mathcal{O}_C) \times \mathrm{Spd}(\mathbb{Q}_p, \mathbb{Z}_p)\,?$$

We will not answer this exact question, but we will see in the next section that the answer to a very similar question is "yes".

15.4. **Prismatic cohomology.** We will give an overview of prismatic cohomology. Let C be a complete algebraically closed extension of $\mathbb{Q}_p$, and let $\mathcal{O}_C$ be its ring of integers. Let X be a smooth formal scheme over $\mathcal{O}_C$. We have seen that the p-adic étale cohomology and de Rham cohomology of the generic fiber and the crystalline cohomology of the special fiber are all closely related. There is cohomology theory called prismatic cohomology that specializes to all three of these. It has coefficients in $A_{\inf}$.

I will define prisms later in the lecture, but for now, let me just say that a prism consists of pair (A, I), where A is a ring with additional structure and $I \subset A$ is an ideal satisfying certain properties. The example that you should keep in mind for now is $A = W(\mathcal{O}_{K^\flat})$, $I = \ker\Theta$. The additional structure includes a Frobenius map ϕ_A on A.

For any perfect field k, $A = W(k)$, $I = (p)$ is also an example of a prism.

Both of these examples are "perfect" and "bounded".

Given a bounded prism (A, I) and a smooth formal scheme X over A/I, one can define the prismatic site $((X/A)_{\mathbb{\Delta}}, \mathcal{O}_{\mathbb{\Delta}})$. One can then define the prismatic cohomology

$$R\Gamma_{\mathbb{\Delta}}(X/A) := R\Gamma((X/A)_{\mathbb{\Delta}}, \mathcal{O}_{\mathbb{\Delta}})\,.$$

It is an object in the derived category of A-modules, and it is equipped with a ϕ_A-linear map ϕ.

Theorem 15.4.1.

(1) *(Crystalline comparison) If $I = (p)$, then there is a canonical ϕ-equivariant isomorphism*

$$R\Gamma_{\mathrm{cris}}(X/A) \cong \phi_A^* R\Gamma_{\mathbb{\Delta}}(X/A)\,.$$

(2) *(de Rham comparison) There is a canonical isomorphism*

$$R\Gamma_{\mathrm{dR}}(X/(A/I)) \cong R\Gamma_{\mathbb{\Delta}}(X/A) \hat{\otimes}^L_{A,\phi_A} A/I\,.$$

(3) *(Étale comparison) Assume A is perfect. Let X_η be the generic fiber of X. For any $n \geq 0$, there is a canonical isomorphism*

$$R\Gamma_{\text{ét}}(X_\eta, \mathbb{Z}/p^n\mathbb{Z}) \cong \left(R\Gamma_{\Delta}(X/A)/p^n[1/I]\right)^{\phi=1} .$$

(4) *(Base change) Let $(A, I) \to (B, J)$ be a map of bounded prisms, and let $Y = X \times_{\mathrm{Spf}(A/I)} \mathrm{Spf}(B, J)$. Then there is a canonical isomorphism*

$$R\Gamma_{\Delta}(X/A)\hat{\otimes}^L_A B \cong R\Gamma_{\Delta}(Y/B) .$$

Remark 15.4.2. In particular, (4) can be applied with $A = W(\mathcal{O}_{K^\flat})$, $I = \ker \Theta$, $B = W(k)$, $I = (p)$, where k is the residue field of K. Combined with (1), we obtain a comparison of the prismatic cohomology of X and with the crystalline cohomology of its special fiber.

16. Prismatic cohomology: examples and definition

16.1. Examples.

Example 16.1.1. Let K be a perfectoid field. Let k be the residue field of K. Suppose $A = W(\mathcal{O}_{K^\flat})$, $I = \ker \Theta$, $X = \mathrm{Spf}(A/I) = \mathrm{Spf}(\mathcal{O}_K)$. Then $R\Gamma_{\Delta}(X/A)$ is just A in degree 0.

If $\mathbb{Q}_p^{\mathrm{cyc}} \subseteq K$, define

$$A\{1\} := \phi_A^{-1}(\mu)A .$$

where $\mu = [\epsilon] - 1$. (To make this construction functorial, we should with $\mathbb{Z}_p(-1)$, but for simplicity, we will ignore this issue.) We will compare the crystalline, étale, and de Rham specializations of A and $A\{1\}$. We may as well take the Frobenus twist now; we have $\phi_A^* A \cong A$, $\phi_A^* A\{1\} \cong \mu A$.

The crystalline specialization of A is $W(k)$ with the usual Frobenius action. The crysatlline specialization of $A\{1\}$ is a free $W(k)$-module of rank 1 generated by μ. The Frobenius sends $\mu \mapsto p\mu$ since the image of $\phi(\mu)/\mu = \frac{[\epsilon]^p - 1}{[\epsilon]-1} = 1 + [\epsilon] + \cdots + [\epsilon]^{p-1}$ in $W(k)$ is p.

The étale specialization of A is

$$\left(A \otimes_A W(K^\flat)/p^n\right)^{\phi=1} \cong W(\mathbb{F}_p)/p^n \cong \mathbb{Z}/p^n\mathbb{Z} .$$

Since $\epsilon - 1 \neq 0$, μ is a unit in $W(K^\flat)$, we again get $\mathbb{Z}/p^n\mathbb{Z}$ for the étale specialization.

The de Rham specialization in each case is a free $\mathcal{O}_C$-module of rank 1 We recover the Hodge filtration as follows. We have

$$A \otimes_A B_{\mathrm{dR}} \cong A\{1\} \otimes_A B_{\mathrm{dR}} \cong \mathbb{Z}_p \otimes_{\mathbb{Z}_p} B_{\mathrm{dR}}$$

where $\mathbb{Z}_p$ is the inverse limit of the étale specializations. We take the usual filtration on $\mathbb{Z}_p \otimes_{\mathbb{Z}_p} B_{\mathrm{dR}}$ and pull it back to $A \otimes_A B_{\mathrm{dR}}^+$ and $A\{1\} \otimes_A B_{\mathrm{dR}}^+$. We see that $A \otimes_A B_{\mathrm{dR}}^+ = \mathrm{Fil}^0$ and $A\{1\} \otimes_A B_{\mathrm{dR}}^+ = \mu B_{\mathrm{dR}}^+ = \mathrm{Fil}^1$.

16.2. Consequences.

Theorem 16.2.1. *Let C be an algebraically closed nonarchimedean field of mixed characteristic, and let k be its residue field. Let X be a proper smooth formal scheme over $\mathcal{O}_C$. Then*

$$\dim_k H^i_{\mathrm{dR}}(X_k) \geq \dim_{\mathbb{F}_p} H^i_{\text{ét}}(X_C, \mathbb{F}_p) .$$

Proof. We have

$$R\Gamma_{\text{ét}}(X_C, \mathbb{F}_p) \cong \left(R\Gamma_{\Delta}(X/A_{\text{inf}}) \otimes^L_{A_{\text{inf}}} C^\flat\right)^{\phi=1}$$

Since $C^\flat$ is separably closed, every étale ϕ-module over $C^\flat$ is trivial. So

$$R\Gamma_{\text{ét}}(X_C, \mathbb{F}_p) \otimes^L_{\mathbb{F}_p} C^\flat \cong R\Gamma_{\Delta}(X/A_{\text{inf}}) \otimes^L_{A_{\text{inf}}} C^\flat .$$

On the other hand,

$$R\Gamma_{\text{dR}}(X_k) \cong R\Gamma_{\Delta}(X/A_{\text{inf}}) \otimes^L_{A_{\text{inf}}} k .$$

The proof is then analogous the proof in algebraic topology that

$$\dim_{\mathbb{F}_p} H^i(Y, \mathbb{F}_p) \geq \dim_{\mathbb{Q}} H^i(Y, \mathbb{Q})$$

for a real manifold Y. Here $\mathbb{F}_p$ and $\mathbb{Q}$ are replaced by k and $C^\flat$, respectively. See [BMS18, Thm. 14.5(ii)] for details. □

16.3. **Definition of prisms.** Let A be a ring. Let $\phi\colon A \to A$ be an endomorphism of A lifting the Frobenius on A/pA. Then $\phi(x) = x^p + p\delta(x)$ for some function $\delta\colon A \to A$. An arbitrariy chosen δ will not necessarily give us an endomorphism of A. If we impose the following constraints on δ, then we will get an endomorphism.

$$\delta(0) = \delta(1) = 0 \tag{16.3.1}$$

$$\delta(x+y) = \delta(x) + \delta(y) - \sum_{n=1}^{p-1} \frac{(p-1)!}{n!(p-n)!} x^n y^{p-n} \tag{16.3.2}$$

$$\delta(xy) = y^p \delta(x) + x^p \delta(y) + p\delta(x)\delta(y) \tag{16.3.3}$$

Definition 16.3.4. A δ-ring is a pair (A, δ) where A is a commutative ring and $\delta\colon A \to A$ is a map of sets satisfying equations (16.3.1–16.3.3).

Remark 16.3.5. If p is not a zero divisor in A, then δ-ring structures on A are in bijection with lifts of Frobenius on A. On the other hand, the identity $\delta(p^n) = p^{n-1} - p^{pn-1}$ shows that if p is nilpotent in A (and A is not the zero ring), then A has no δ-ring structures.

Definition 16.3.6. A *prism* is a pair (A, I) where A is a δ-ring and I is an ideal of A such that A is derived (p, I)-complete, and $p \in I + \phi(I)A$.

We will not attempt to define what it means for a ring to be derived (p, I)-complete. However, "complete" implies "derived complete" and "derived complete and separated" implies "complete".

Definition 16.3.7. A prism is *perfect* if the Frobenius map $\phi\colon A \to A$ is an isomorphism.

A prism is *bounded* if it has bounded p^∞-torsion, i.e. $A[p^\infty] = A[p^n]$ for some n.

Example 16.3.8.

(1) As mentioned before, for any perfectoid field K, $(W(\mathcal{O}_{K^\flat}), \ker\Theta)$ is a prism with $A/I \cong \mathcal{O}_K$, and for any perfect field k, $(W(k), (p))$ is a prism with $A/I \cong k$. These are perfect and bounded.

(2) Let $A = \mathbb{Z}_p[\![T]\!]$ with Frobenius $T \mapsto T^p$, and let $I = (T - p)$. Then (A, I) is a bounded prism with $A/I \cong \mathbb{Z}_p$.

(3) Let $A = \mathbb{Z}_p[\![T]\!]$ with Frobenius $T \mapsto (1+T)^p - 1$, and let $I = \left(\frac{(1+T)^p - 1}{T}\right)$. Then (A, I) is a bounded prism with $A/I \cong \mathbb{Z}_p[\zeta_p]$.

16.4. **The prismatic site.**

Lemma 16.4.1. *Let (A, I) be a prism, and let $J \subset I$ be an ideal of A such that (A, J) is a prism. Then $J = I$.*

Corollary 16.4.2. *If $(A, I) \to (B, J)$ is a map of prisms, then $J = IB$.*

Definition 16.4.3. Let (A, I) be a bounded prism. Let X be a smooth p-adic formal scheme over A/I. Define $(X/A)_{\mathbb{\Delta}}$ to be the category of maps $(A, I) \to (B, IB)$ of bounded prisms, together with a map $\mathrm{Spf}(B/IB) \to X$ over A/I.

A morphism in $(X/A)_{\mathbb{\Delta}}$ is a *flat cover* if the induced map of prisms $(B, IB) \to (C, IC)$ is faithfully flat, i.e. C is (p, IB)-completely flat over B ($C/(p, I)C$ is a flat $B/(p, I)B$-module and $\mathrm{Tor}^n_B(C, B/(p, I)B) = 0$ for $n > 0$).

The *prismatic site* of X/A is the category $(X/A)_{\mathbb{\Delta}}$ along with the topology defined by flat covers. Define sheaves $\mathcal{O}_{\mathbb{\Delta}}$, $\overline{\mathcal{O}}_{\mathbb{\Delta}}$ on $(X/A)_{\mathbb{\Delta}}$ by

$$\mathcal{O}_{\mathbb{\Delta}}(\mathrm{Spf}(B) \hookleftarrow \mathrm{Spf}(B/IB) \to X) = B$$
$$\overline{\mathcal{O}}_{\mathbb{\Delta}}(\mathrm{Spf}(B) \hookleftarrow \mathrm{Spf}(B/IB) \to X) = B/IB$$

16.5. **Vector bundles on the Fargues–Fontaine curve and cohomology.** It would seem that a similar construction might be possible on the Fargues–Fontaine curve. Let Y be a proper smooth rigid space over C. Then one could try constructing a site where the objects are diagrams

$$\begin{array}{ccc} Z \times_{\mathcal{X}_{C^\flat}} x_C & \hookrightarrow & Z \\ \downarrow & & \downarrow \\ Y & & \mathcal{X}_{C^\flat}\,. \end{array}$$

I don't know exactly what sort of morphisms and coverings should be allowed, though.

Now I will mention a similar cohomology theory for quasicompact rigid spaces over $\mathrm{Spa}(C, \O_C)$. It takes values in vector bundles over the Fargues–Fontaine curve $X_{C^\flat}$.

Note that we do not require properness. The étale cohomology of non-proper rigid spaces is generally huge. We saw in an earlier lecture that the closed unit disc is not simply connected; in fact it has many Artin-Schreier $\mathbb{Z}_p$-covers. Taking the de Rham cohomology of a non-proper rigid space is also tricky, as the antiderivative of a power series converging on the unit disc does not necessarily converge on the unit disc. To deal with this problem, one considers the *overconvergent de Rham complex* $\Omega^{\bullet,\dagger}$. For example, a function on the closed unit disc is called overconvergent if it converges on some larger disc. The antiderivative of an overconvergent function is then overconvergent.

Theorem 16.5.1. *There is a cohomology thery $\mathcal{FF}$ on the category of quasicompact, separated, taut rigid spaces to $D^b(\mathrm{Coh}_{X_{C^\flat}})$ satisfying the following properties:*

1. *$H^i_{\mathcal{FF}}(Z) = 0$ for $i < 0$, $i > 2 \dim Z$*
2. *If Z is defined over a p-adic field K and is proper and smooth, then $H^i_{\mathcal{FF}}(Z_C)$ is the G_K-equivariant vector bundle associated with the G_K representation $H^i_{\text{ét}}(Z_C, \mathbb{Q}_p)$.*
3. *If Z is defined over a p-adic field K, then the completion of the stalk of $H^i_{\mathcal{FF}}(Z_C)$ at x_C is isomorphic to $H^i_{\mathrm{dR}}(Z/K)^\dagger \otimes_K B^+_{\mathrm{dR}}$.*

(4) *If* $\mathfrak{Z}$ *is a proper smooth formal scheme over* $\mathcal{O}_C$ *with generic fiber* Z *and special fiber* $\mathfrak{Z}_s$*, then* $H^i_{\mathcal{FF}}(Z_C)$ *is the vector bundle associated with the isocrystal* $H^i_{\mathrm{cris}}(\mathfrak{Z}_s) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$.

References

[Ax70] J. Ax. Zeros of polynomials over local fields—The Galois action. *J. Algebra*, 15:417–428, 1970.

[BC] O. Brinon and B. Conrad. CMI summer school notes on p-adic Hodge theory. `http://math.stanford.edu/~conrad/papers/notes.pdf`.

[BMS18] B. Bhatt, M. Morrow, and P. Scholze. Integral p-adic Hodge theory. *Publ. Math. Inst. Hautes Études Sci.*, 128:219–397, 2018.

[Dia17] H. Diao. Period rings and period sheaves. 2017. `http://swc.math.arizona.edu/aws/2017/2017DiaoProblems.pdf`.

[Far08] L. Fargues. L'isomorphisme entre les tours de Lubin-Tate et de Drinfeld et applications cohomologiques. In *L'isomorphisme entre les tours de Lubin-Tate et de Drinfeld*, volume 262 of *Progr. Math.*, pages 1–325. Birkhäuser, Basel, 2008.

[FF18] L. Fargues and J.-M. Fontaine. Courbes et fibrés vectoriels en théorie de Hodge p-adique. *Astérisque*, (406):xiii+382, 2018. With a preface by Pierre Colmez.

[FO] J.-M. Fontaine and Y. Ouyang. Theory of p-adic Galois representations. `http://staff.ustc.edu.cn/~yiouyang/galoisrep.pdf`.

[Fon82] J.-M. Fontaine. Sur certains types de représentations p-adiques du groupe de Galois d'un corps local; construction d'un anneau de Barsotti-Tate. *Ann. of Math. (2)*, 115(3):529–577, 1982.

[Fon82] J.-M. Fontaine. Formes différentielles et modules de Tate des variétés abéliennes sur les corps locaux. *Invent. Math.*, 65(3):379–409, 1981/82.

[Kat73] N. M. Katz. p-adic properties of modular schemes and modular forms. In *Modular functions of one variable, III (Proc. Internat. Summer School, Univ. Antwerp, Antwerp, 1972)*, pages 69–190. Lecture Notes in Mathematics, Vol. 350. Springer, Berlin, 1973.

[Ked15] K. S. Kedlaya. New methods for (Γ, φ)-modules. *Res. Math. Sci.*, 2:Art. 20, 31, 2015.

[Mes72] W. Messing. *The crystals associated to Barsotti-Tate groups: with applications to abelian schemes*. Lecture Notes in Mathematics, Vol. 264. Springer-Verlag, Berlin-New York, 1972.

[Sta] Stacks Project Authors. Stacks Project. `http://stacks.math.columbia.edu`.

[SW13] P. Scholze and J. Weinstein. Moduli of p-divisible groups. *Camb. J. Math.*, 1(2):145–237, 2013.

[Tat67] J. T. Tate. p-divisible groups. In *Proc. Conf. Local Fields (Driebergen, 1966)*, pages 158–183. Springer, Berlin, 1967.

[Tat97] J. Tate. Finite flat group schemes. In *Modular forms and Fermat's last theorem (Boston, MA, 1995)*, pages 121–154. Springer, New York, 1997.