p-adic Hodge theory homework: Week 9

1. Let K be a field. Compute the de Rham cohomology of $\mathbb{P}^1_K$.

2. Let $\mathbb{G}_m$ be the contravariant functor that sends a scheme X to the abelian group $\mathcal{O}_X(X)^\times$. It is possible to show that $\mathbb{G}_m$ is a sheaf for both the Zariski and the étale topology.

   By [Sta, Tag 03P7], there are isomorphisms

   $$\mathrm{Pic}(X) \cong H^1(X_{\mathrm{Zar}}, \mathbb{G}_m) \cong H^1(X_{\text{ét}}, \mathbb{G}_m)\,. \qquad (*)$$

   (See tags 040E and 03AJ for key ideas in the proof.) Here, $\mathrm{Pic}(X)$ denotes the group of isomorphism classes of line bundles on X, up to isomorphism.

   Suppose we are working over a field K, and let n be an integer that is invertible in K. Let $[n]\colon \mathbb{G}_m \to \mathbb{G}_m$ denote the nth power map. Let $\mu_n = \ker[n]$.

   (a) Explain why $[n]$ is a surjection of étale sheaves, but not a surjection of Zariski sheaves.

   (b) Use (*) along with the exact sequence of étale sheaves

   $$1 \to \mu_n \to \mathbb{G}_m \xrightarrow{[n]} \mathbb{G}_m \to 1$$

   to compute
   $$H^1_{\text{ét}}(\mathbb{A}^1_K \setminus \{0\}, \mu_n)\,.$$

   (c) Let E be an elliptic curve over K. Show that there is a canonical isomorphism
   $$H^1_{\text{ét}}(E, \mu_n) \cong E(K)[n]\,.$$

   Note that if K contains an nth root of unity, then μ_n is (noncanonically) isomorphic to the constant sheaf $\mathbb{Z}/n\mathbb{Z}$.

3. Let K be a p-adic field, and let $M/L/K$ be a tower of finite extensions. Show that the natural map

   $$\Omega_{\mathcal{O}_L/\mathcal{O}_K} \otimes_{\mathcal{O}_L} \mathcal{O}_M \to \Omega_{\mathcal{O}_M/\mathcal{O}_K}$$

   is injective. (Hint: one way to prove this is to use cotangent complexes. If you are not familiar with cotangent complexes, I suggest reading section 2.1 of `https://arxiv.org/pdf/1606.01921` and/or section 3.1 of `https://swc-math.github.io/aws/2017/2017BhattNotes.pdf`.)