p-adic Hodge theory homework: Week 3

1. [Sil] exercise 7.5a.

2. Verify that for any $a, b \in \mathbb{Q}_2$,
$$|a^2 + b^2|^{1/2} = \max(|a+b|, |2|^{1/2}|b|)\,,$$
and that the function
$$a + bi \mapsto |a^2 + b^2|^{1/2}$$
defines an absolute value on $\mathbb{Q}_2(i)$. Verify that the residue field of $\mathbb{Q}_2(i)$ is $\mathbb{F}_2$.

3. In this problem, we will use the Néron-Ogg-Shafarevich criterion to show that the elliptic curve E defined by $y^2 = x^3 - x$ does not have good reduction at the prime 2.

   (a) Over the field $\mathbb{Q}_2(i)$, E has an action by the ring $\mathbb{Z}[i]$, where the action of i is given by $(x, y) \mapsto (-x, iy)$. Use the previous exercise to argue that $\mathrm{Gal}(\overline{\mathbb{Q}}_2/\mathbb{Q}_2(i))$ does not contain the inertia subgroup of $\mathrm{Gal}(\overline{\mathbb{Q}}_2/\mathbb{Q}_2)$. In other words, the extension $\mathbb{Q}_2(i)/\mathbb{Q}_2$ is ramified.

   (b) Let ℓ be a prime. Both $\mathrm{Gal}(\overline{\mathbb{Q}}_2/\mathbb{Q}_2)$ and $\mathbb{Z}[i]$ act on $T_\ell(E)$. Argue that for any $\sigma \in \mathrm{Gal}(\overline{\mathbb{Q}}_2/\mathbb{Q}_2)$, $\sigma i = (\sigma(i))\sigma$ as endomorpisms of $T_\ell(E)$.

   (c) Argue that the kernel of the action of $\mathrm{Gal}(\overline{\mathbb{Q}}_2/\mathbb{Q}_2)$ on $T_\ell(E)$ must be contained in $\mathrm{Gal}(\overline{\mathbb{Q}}_2/\mathbb{Q}_2(i))$. Take $\ell \neq 2$, and use the Néron-Ogg-Shafarevich criterion to conclude that E does not have good reduction.