p-adic Hodge theory homework: Week 13

1. [Mie] problem 4.8.

2. (a) Let K be a perfectoid field of characteristic zero. Let $\widetilde{\mathbb{G}_{m,K}}$ be the functor that sends a perfectoid Huber pair (R, R^+) over $(K, \mathcal{O}_K)$ to $1 + (R^\flat)^{\circ\circ}$. Show that $\widetilde{\mathbb{G}_{m,K}}$ is representable by a perfectoid space.

   (b) Show that the map $\widetilde{\mathbb{G}_{m,K}} \to \mathbb{A}^1_K$ sending $z \in 1 + (R^\flat)^{\circ\circ}$ to $\log(z^\sharp)$ induces a surjection of pro-étale sheaves $(\widetilde{\mathbb{G}_{m,K}})^\flat \to (\mathbb{A}^1_K)^\diamondsuit$.

3. Let C be an algebraically closed perfectoid field of characteristic p.

   (a) Show that $(1 + \mathfrak{m}_C)^\times$ has a natural $\mathbb{Q}_p$-vector space structure.

   (b) Let $C^\sharp$ be a characteristic unzero untilt of C. The $\sharp$ map induces a group homomorphism $(1 + \mathfrak{m}_C)^\times \to (1 + \mathfrak{m}_{C^\sharp})^\times$. Show that the preimage of the group of p-power roots of unity $\mu_{p^\infty}(C^\sharp)$ is a one-dimensional $\mathbb{Q}_p$-vector subspace of $(1 + \mathfrak{m}_C)^\times$.

   (c) Conversely, show for every one-dimensional $\mathbb{Q}_p$-vector subspace V of $(1 + \mathfrak{m}_C)^\times$, there is an untilt $C^\sharp$ such that $\sharp(V) = \mu_{p^\infty}(C^\sharp)$. Further show that $C^\sharp$ is unique up to isomorphism, and the map $(C^\sharp)^\flat \xrightarrow{\sim} C$ is unique up to composing with a power of Frobenius. (Hint: use the fact that $C^\sharp$ must contain $\mathbb{Q}_p^{\mathrm{cyc}}$, and $(\mathbb{Q}_p^{\mathrm{cyc}})^\flat \cong \mathbb{F}_p((t^{1/p^\infty}))$.)