

# Lane Changes and Close Following: Troublesome Tollbooth Traffic

Andrew Spann

Daniel Kane

Dan Gulotta

Massachusetts Institute of Technology  
Cambridge, MA

Advisor: Martin Zdenek Bazant

## Summary

We develop a cellular-automaton model to address the slow speeds and emphasis on lane-changing in tollbooth plazas. We make assumptions about car-following, based on distance and relative speeds, and arrive at the criterion that cars maximize their speeds subject to

$$\text{gap} > \left\lfloor \frac{V_{\text{car}}}{2} \right\rfloor + \frac{1}{2}(V_{\text{car}} - V_{\text{frontcar}})(V_{\text{car}} + V_{\text{frontcar}} + 1).$$

We invent lane-change rules for cars to determine if they can turn safely and if changing lanes would allow higher speed. Cars modify these preferences based on whether changing lanes would bring them closer to a desired type of tollbooth. Overall, our assumptions encourage people to be a bit more aggressive than in traditional models when merging or driving at low speeds.

We simulate a 70-min period at a tollbooth plaza, with intervals of light and heavy traffic. We look at statistics from this simulation and comment on the behavior of individual cars.

In addition to determining the number of tollbooths needed, we discuss how tollbooth plazas can be improved with road barriers to direct lane expansion or by assigning the correct number of booths to electronic toll collection. We set up a generalized lane-expansion structure to test configurations.

Booths should be ordered to encourage safe behavior, such as putting faster electronic booths together. Rigid barriers affect wait time adversely.

Under typical traffic loads, there should be *at least twice as many booths as highway lanes*.

---

*The UMAP Journal* 26 (3) (2005) 251–264. ©Copyright 2005 by COMAP, Inc. All rights reserved. Permission to make digital or hard copies of part or all of this work for personal or classroom use is granted without fee provided that copies are not made or distributed for profit or commercial advantage and that copies bear this notice. Abstracting with credit is permitted, but copyrights for components of this work owned by others than COMAP must be honored. To copy otherwise, to republish, to post on servers, or to redistribute to lists requires prior permission from COMAP.

## Definitions and Conventions

**Car/Driver.** Used interchangeably; “cars” includes trucks.

**Tollbooth lane and highway lane.** There are  $n$  highway lanes and  $m$  tollbooths. The tollbooth lane is the lane corresponding to a particular tollbooth after lane expansion.

**Default lane.** In the lane-expansion region, each highway lane is assigned a tollbooth lane such that following the highway lane without turning leads by default to the tollbooth lane, and following that in the lane-contraction region leads to the corresponding highway lane. Other tollbooth lanes begin to exist at the start of lane expansion and become dead ends at the end of lane contraction.

**Delay.** The time for a car to traverse the entire map of our simulated world, which stretches 250 cells before and after the tollbooth.

**Gap.** We represent a lane as an array; the gap is obtained by subtracting the array indices between two adjacent cars.

## Assumptions and Justifications

### Booths

**A booth is manual, automatic, or electronic.** A manual booth has a person who collects the toll, an automatic booth lets drivers deposit coins, and an electronic booth reads a prepaid radio frequency identification tag as the car drives by. A booth may allow multiple types of payment.

**The cost of operating the booths is negligible.** Compared to the cost of building the toll plaza or of maintaining the stretch of highway for which the toll is collected, this expense is insignificant, particularly since automated and electronic booths require less maintenance than manual booths.

**Booth delays.** Cars with an electronic pass can cross electronic tollbooths at a speed of 2 cells per time increment ( $\approx 30$  mph). A car with an electronic pass can also travel through a manual or automatic booth but must wait 3–7 s for the gate to rise. A car without an electronic pass is delayed 8–12 s at an automatic booth and 13–17 s at a manual booth. We use probabilistic uniform distributions over these intervals to ensure that cars do not exit tollbooths in sync.

## Cars/Drivers

**Cars are generated according to a probability distribution.** We start them 1 mi from the tollbooth and generate for a fixed amount of simulated time (usually about 1 h), then keep running until all have gotten to the end of the simulated road 1 mi beyond the tollbooth. There are no entry or exit ramps in the 1 mi section leading to the tollbooth. Some vehicles are classified as trucks, which function identically but must use manual tollbooths if they do not have an electronic pass.

**Drivers accurately estimate distances and differences in speed.**

**Car acceleration and deceleration is linear and symmetric.** In reality, a car can accelerate much faster from 0 to 15 mph than from 45 to 60 mph, and the distances for braking and acceleration are different; but this is a standard assumption for cellular-automaton models.

**Cars pack closely in a tollbooth line.** Drivers don't want people from other lanes to cut into their line, so they follow at distances closer than suggested on state driver's license exams.

**Dissatisfaction from waiting in line is a nondecreasing convex function.** An especially long wait is a major annoyance. In other words, it is better to spread wait times uniformly than to have a high standard deviation.

## Lanes

**The toll collectors can set up new rigid barriers in the lane-expansion region.**

Doing so would make certain lane changes illegal in designated locations. Since adding an extra tollbooth can be cost-prohibitive, setting up barriers to promote efficient lane-splitting and merging is important.

**Signs are posted telling drivers what types of payment each lane accepts.** If drivers benefit from a certain type of booth (e.g., electronic), they will tend to gravitate toward it.

**No highway lane is predisposed to higher speeds than others.** Which lanes are "fast" or "slow" is dictated by the types of tollbooths that they most directly feed into.

**The lane-expansion region covers about 300 ft.** The lane contraction section is also assumed to be 300 ft.

## Criteria for Optimal Tollbooth Configuration

Cars slowing or stopping at tollbooths makes for bottlenecks. Since the speed of a car through a tollbooth must be slower than highway speed, adding tollbooths is an intuitive way to compensate.

Our goal is a configuration of lanes and tollbooths that minimizes delay for drivers. Mean wait time is the simplest criterion but not the best. Consider the case where there are no electronic passes and traffic is very heavy. In this limiting but plausible case, there are constantly lines and cars pass through each booth at full capacity. For a fixed number of tollbooths, the total wait time should be similar regardless of tollbooth-lane configuration; but if one lane is moving notably faster than the others, then the distributions of wait times will differ. Because we assume that dissatisfaction is a convex function, we give more weight to people who are stuck a long time. Klodzinski and Al-Deek [2002, 177] suggest that the 85th percentile of delays is a good criterion

At the same time, we do not wish to ignore drivers who go through quickly. Therefore, we take the mean of the data that fall between the 50th and 85th percentiles for each type of vehicle. This will put an emphasis on cars that are stuck during times of high traffic but will not allow outliers to hijack the data. We consider separately the categories of cars, trucks, and vehicles with electronic passes, take the mean of the data that fall between the 50th and 85th percentiles, and take the weighted average of this according to the percentage of vehicles in the three categories.

We also wish to analyze effect of toll plaza layout. We therefore record the incidence of unusually aggressive lane changes, excessive braking, and cars getting “stuck” in the electronic lane that do not have an electronic pass.

## Setting Up a Model

In the Nagel-Schreckenberg cellular-automaton model of traffic flow [Nagel and Schreckenberg 1992, 2222], cars travel through cells that are roughly the size of a car with speeds of up to 5 cells per time increment. This model determines speeds with the rules that cars accelerate if possible, slow down to avoid other cars if needed, and brake with some random probability. The model updates car positions in parallel. Such models produce beautiful simulations of general highway traffic, but less research has been done using the tight speed constraints and high emphasis on lane-changing that a tollbooth offers.

Creating models for multiple lanes involves defining lane-change criteria, such as change lanes if there is a car too close in the current lane, if changing lanes would improve this, if there are no cars within a certain distance back in the lane to change into, and if a randomly generated variable falls within a certain range [Rickert et al. 1996, 537]. Even with only two lanes, one gets interesting behavior and flow-density relationships that match empirical observations [Chowdhury et al. 1997, 423]. Huang and Huang even try to implement tollbooths into the Nagel-Schreckenberg model, but their treatment of lane expansion assumes that each lane branches into two (or more) tollbooths dedicated to that lane [2002, 602]. In real life, sometimes highways add tollbooth lanes without distributing the split evenly among highway lanes.

To allow for a various setups, we develop a generalized lane-expansion

structure. In the tollbooth scenario, low speeds are more common than in general stretches of highway, and there is a need to address more than two lanes. We change car-following and lane-change rules to fit a congested tollbooth area.

In the Nagel-Schreckenberg model, cars adjust their speeds based on the space in front. The tollbooth forces a universal slowdown in traffic. At these slow speeds, it is possible to follow cars more closely than at faster speeds. In the real world, we consider the speed of the car in front in addition to its distance away. Random braking is needed in the Nagel and Schreckenberg model to prevent the cars from reaching a steady state. However, in a tollbooth scenario, the desire not to let other cars cut in line predisposes drivers to follow the car in front more closely than expected. Thus, we do not use random braking but incorporate randomness into the arrival of new cars and lane-change priority orders. Instead of Nagel and Schreckenberg's rules, we propose the following rules for a cellular automaton model simulating a tollbooth scenario:

1. Cars have a speed of from 0 to 5 cells per time increment. In a single time increment, they can accelerate or decelerate by at most 1 unit.
2. Drivers go as fast as they can, subject to the constraint that the distance to the car in front is enough so that if it brakes suddenly, they can stop in time.
3. Cars change lanes if doing so would allow them to move faster. They modify the increased speed benefits of changing lanes by checking if the lane leads to a more desirable tollbooth type or if they face an impending lane merger. Before changing lanes, cars check the gap criterion of rule 2) applies to both the gap in front and the gap behind the driver after the lane change.
4. At each time step, we update positions and speeds from front to back.

We examine the rules in detail. Let us say that

- There are 250 cells in a mile (a little over 21 ft/cell).
- Each time step represents about 1 s.

Rule 1's maximum speed of 5 cells per time step corresponds to 72 mph (each unit of velocity is just under 15 mph), which is about the expected highway speed. The numbers for length and time increments do not need to be precise, since we can fix one and scale the other; what is important is that the length of a cell is a little more than the average length of a car (about 15 ft).

For any pair of following cars, we want the rear car to be able to decelerate at a rate of at most 1 unit and still avoid collision with the front car, even if the front car begins decelerating at a rate of 1 unit per time step squared. If for a given time step the rear and front cars have speeds  $V_{\text{car}}$  and  $V_{\text{frontcar}}$  and immediately begin decelerating at a rate of 1 unit squared until they stop, the total distances that they travel are

$$V_{\text{car}} + (V_{\text{car}} - 1) + \cdots + 1 = \frac{1}{2} V_{\text{car}} (V_{\text{car}} + 1),$$

$$V_{\text{frontcar}} + (V_{\text{frontcar}} - 1) + \cdots + 1 = \frac{1}{2} V_{\text{frontcar}} (V_{\text{frontcar}} + 1).$$

Our condition is equivalent to the car in back remaining behind the car in front, so the gap or difference in squares between the cars must be

$$\text{gap} > \frac{1}{2}V_{\text{car}}(V_{\text{car}}+1) - \frac{1}{2}V_{\text{frontcar}}(V_{\text{frontcar}}+1) = \frac{1}{2}(V_{\text{car}} - V_{\text{frontcar}})(V_{\text{car}} + V_{\text{frontcar}} + 1).$$

Thus, at each update, the rear car checks if it can increase its speed by 1 and still satisfy this inequality or if it must decrease its speed to maintain the inequality, and acts accordingly.

However, according to this model, if two cars are going the same speed, then they theoretically touch. Besides being a safety problem, this also contradicts the observations of Hall et al. that flow of cars as a function of percent occupancy of a location increases sharply until about 20% and then decreases thereafter [1986, 207]. With the inequality above, we could generate initial conditions with high occupancy and high flow. Before we discard our model, though, let us first check to see if these conditions would actually show up in the simulation.

We add the rule that a car tries to leave at least  $\lfloor V_{\text{car}}/2 \rfloor$  empty spaces before the car in front; this would still let cars tailgate at low speeds. For high speeds, this would be a somewhat unsafe distance but consistent with aggressive merging; but we expect high speeds to be rare near the tollbooth during moderate or high congestion. Thus, our final criterion for rule 2 is that a car looks at the number of empty spaces in front of it and adjusts its speed (upward if possible) so that it still meets the inequality

$$\text{gap} > \left\lfloor \frac{V_{\text{car}}}{2} \right\rfloor + \frac{1}{2}(V_{\text{car}} - V_{\text{frontcar}})(V_{\text{car}} + V_{\text{frontcar}} + 1).$$

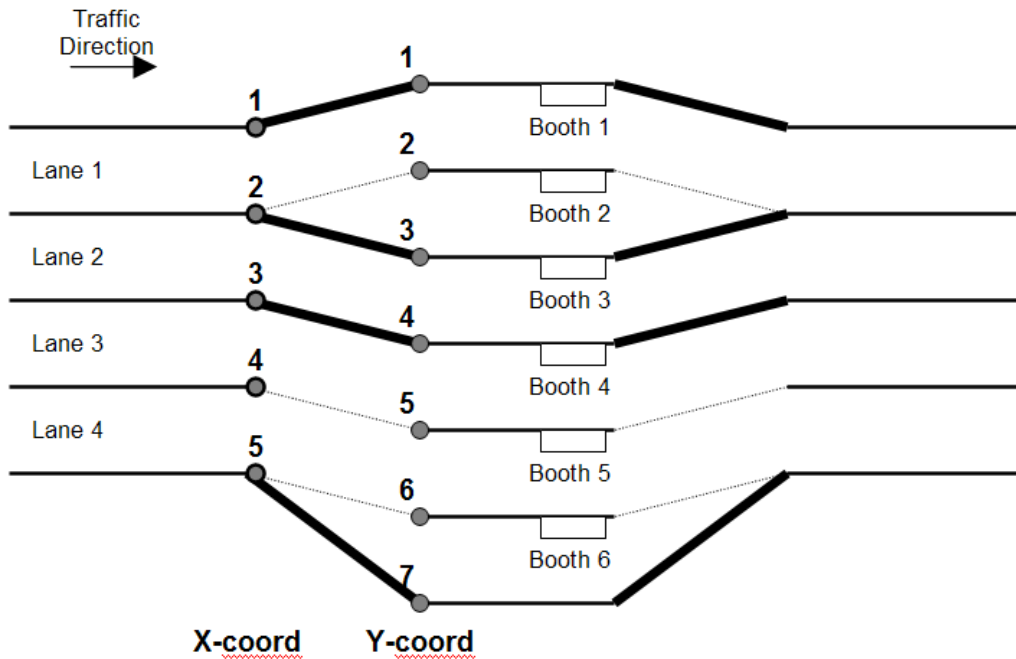
When changing lanes (rule 3), cars ask, “If I changed lanes, how fast could I go this time step?” Cars avoid making lane changes that could cause a collision, as determined by the gap criterion. When given an opportunity to change lanes, a car compares the values of the maximum speeds that it could attain if it were in each lane but adds modifiers. Lanes have penalties in valuation for leading to tollbooths that the driver cannot use (−2 per lane away from a usable lane) or are suboptimal (−1 per lane away from an optimal lane), where suboptimal means a car with an electronic pass in a lane that does not accept it. Leaving the tollbooth, drivers try hard to get out of dead-end lanes (−3 or −5 depending on how far it is to the end). If drivers value the lane they are in and a separate lane equally, they do not change. If drivers value both the lane on their left and the lane on their right equally more than their current lane, they pick randomly.

We update speeds in each lane from front to back, with lanes chosen in a random order. A consequence of this is that information can propagate backwards at infinite speed if for example the head of a string of cars all going at speed 1 came to a complete stop. This infinite wave-speed problem could be fixed by introducing random braking, but at slow speeds we find it more acceptable to have people inching forward continuously than to have people braking from 1 to 0 at inopportune times. This also has consequences for lane changes, in that randomly giving lanes an update priority will have different results from processing all lane changes in parallel. Although some cellular-automaton traffic

models in the literature update in parallel, we use serial updating because it makes handling the arrays easier and eliminates the problem of having people from two different lanes trying to change into the lane between them at the same time. The random update priority ordering for the lanes is changed every time increment, so that there is less systematic asymmetry in lane changing.

## Generalized Lane-Expansion Structure

We develop a system to describe easily a large number of different tollbooth setups. The road both starts and ends as an  $n$ -lane highway and contains  $m$  tollbooths in the middle. The lane dividers are labeled from 1 to  $n + 1$  for the highway lanes and from 1 to  $m + 1$  for the tollbooth lanes. A rigid barrier consists of an  $(x, y)$  pair where the  $x$ -coordinate represents a lane divider and the  $y$ -coordinate represents a tollbooth divider. **Figure 1** shows the case  $n = 4$ ,  $m = 6$ , with rigid barriers at  $(1, 1)$ ,  $(2, 3)$ ,  $(3, 4)$ , and  $(5, 7)$ . Cars may not make lane changes across a rigid barrier.



**Figure 1.** Generalized lane-expansion scheme.

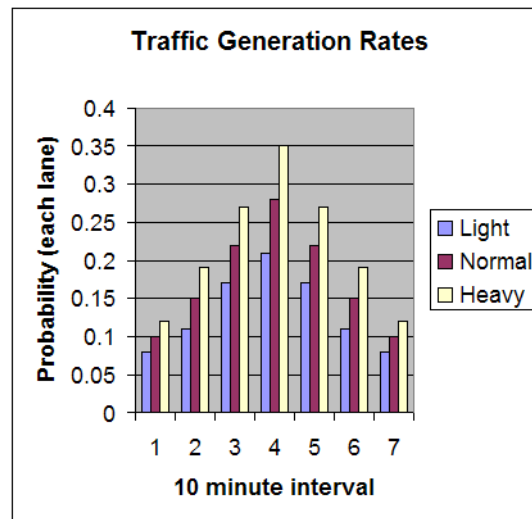
Suppose that we have an ordered set of rigid barriers  $\{(x_1, y_1), \dots, (x_k, y_k)\}$ , where  $x_{i+1} > x_i$  for  $i = 1, \dots, k-1$ . Then the set of rigid barriers must obey the following rules:

- you cannot drive off the road:  $(1, 1)$  and  $(n+1, m+1)$  must be rigid barriers; and
- rigid barriers do not cross each other: for  $i = 1, \dots, k-1$ , we have  $y_{i+1} > y_i$ .

The dotted lines in **Figure 1** can be crossed as normal lane changes. In the lane-expansion region, each lane is assigned a “default” tollbooth lane that it most naturally feeds into. Highway lanes 1, 2, 3, and 4 feed tollbooth lanes 1, 3, 4, and 5. If there is an adjacent lane not blocked by a rigid barrier, a car can enter that lane. The default tollbooth lane then feeds back into the highway lane after the tollbooth and the other highway lanes are treated as dead-ends. Cars are given an incentive to change out of these dead-end lanes ahead of time. We assume that rigid barriers and default lanes are symmetric between lane expansion and contraction. Additionally, no lane changes are allowed on the five cells immediately preceding and following the tollbooth cell.

## Results

We simulate a 70-min period when incoming traffic starts light, increases for 40 min, then decreases again. **Figure 2** shows the generation rates for light, normal, and heavy traffic. For a four-lane highway, these settings correspond to volumes of about 2200, 3000, and 3600 cars over the 70-min period.

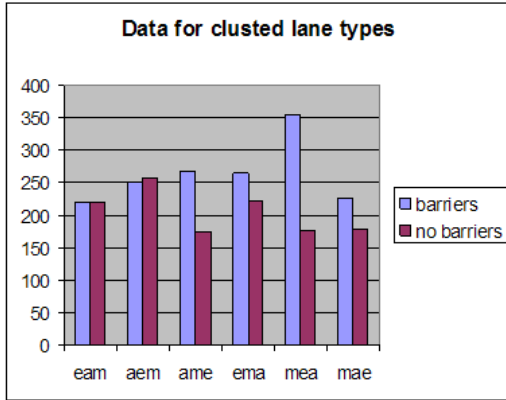


**Figure 2.** Traffic generation rates.

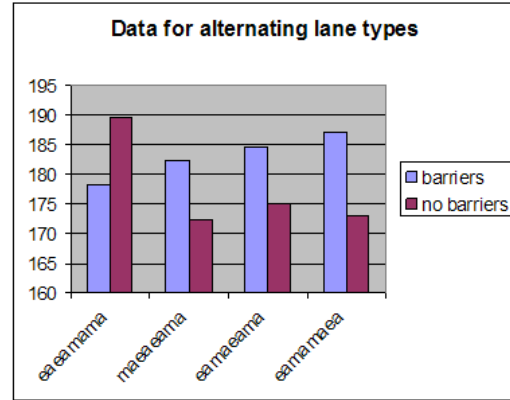
We test two cases of allocation and arrangement of tollbooths: no barriers, or else each highway lane branches into an equal number of tollbooth lanes. In both cases, we make the odd numbered lanes the default lanes. We tested both of these cases for different orderings of the tollbooths. For a 4-lane highway, we use 2 electronic booths, 4 automatic booths, and 2 manual booths. Half of the vehicles had electronic passes and 10% of the vehicles are trucks (no electronic pass). First we clustered all booths of the same type in some permutation, then we alternated types. **Figure 3** shows data averaged over 10 runs.

The  $x$ -axis gives configurations and the  $y$ -axis is adjusted average delay time (mean of the 50th through 85th percentiles of travel time through the 250





**Figure 3a.** Clustered lanes.



**Figure 3b.** Alternating lanes.

**Figure 3.** Average delays for two lane configurations and with vs. without barriers (normal traffic load, 4 highway lanes, 8 booths (2 **e**lectronic, 4 **a**utomatic, 2 **m**anual)).

cells before and after the tollbooth). If there were no tollbooth, then a car at full speed would have delay 100 s. Barriers are slightly worse than just allowing people to change lanes.

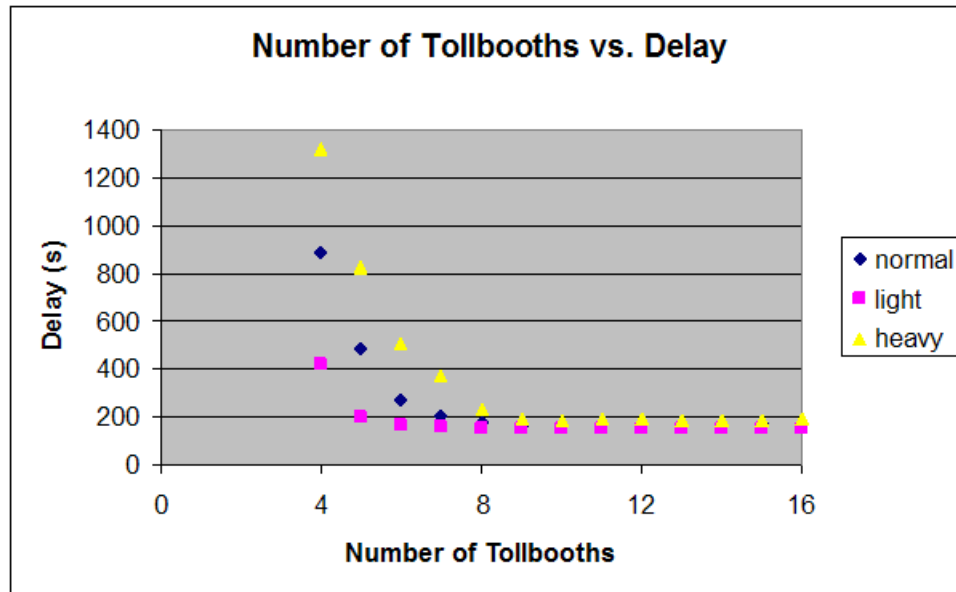
We note from **Figure 3a** that each of the clustered lane types is different from its mirror image, and this phenomenon is reproducible, which is puzzling. We think that it is caused by our handling of the concept of default lane, where some lanes feed directly into tollbooth lanes; with unrestricted lane expansion, this might make some lane changes easier than others.

The alternating tollbooth configurations appear to have slightly less delay, but they also generate more warning flags about dangerous turns and cars becoming stuck in tollbooth lanes that they cannot use. For each clustered configuration, either it or its mirror image gives time equivalent to the alternating tollbooth configurations. Therefore, for safety reasons, we suggest using the clustered configurations.

We next determine how many of each type of booth to use for a 4-lane highway with 8 tollbooths and a lane-expansion region with no rigid barriers. We put the electronic booths on the left and the manual booths on the right. We vary the numbers of each type of tollbooth for different distributions of cars, trucks, and vehicles with electronic passes and run the simulation under normal traffic loads. Unless the percentage of trucks is very low, allocating only 1 manual booth for the trucks generates a large number of trucks stuck in lanes that they cannot use. The number of electronic booths should be 2 or 3, depending on whether cars with electronic passes outnumber vehicles without them. Since 8 to 12 tollbooths is reasonable size for a 4-lane highway, *we recommend very roughly one-fourth electronic, one-half automatic, and one-fourth manual tollbooths.*

How many tollbooths are needed for different levels of traffic? We round down the number of manual and electronic tollbooths and round up the number of automatic tollbooths from the above proportions. Using the light, normal,

and heavy traffic loads defined in **Figure 2** above, we arrive at **Figure 4**.



**Figure 4.** Delay vs. number of tollbooths.

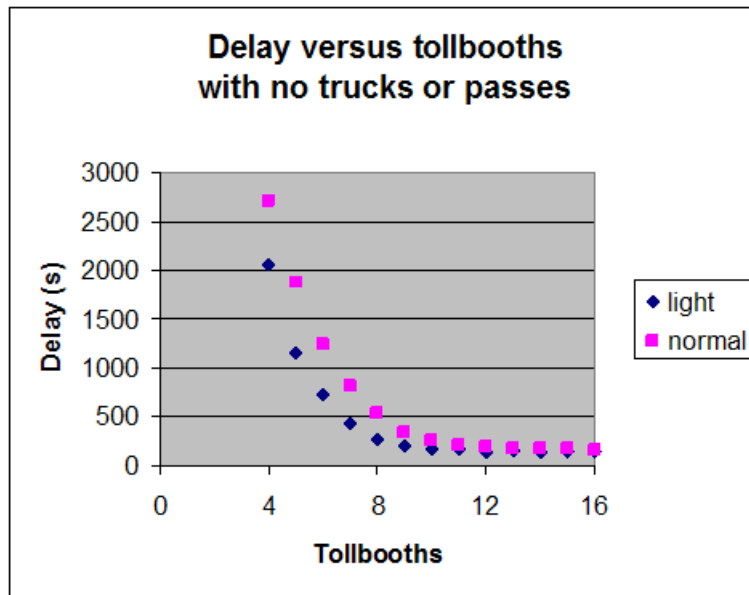
Finally, we consider the limiting case of no trucks and no electronic passes (all tollbooths are automatic). This is the standard case directly comparable to other models in the literature. Under light, normal, and heavy traffic loads, we find that delay times are as in **Figure 5**.

Without electronic tollbooths, cars experience much longer delays, since each must stop at a tollbooth. With only one tollbooth per lane, the normal traffic load (3000 vehicles over 70 min) forces many people to wait over 45 min! It takes about 12 lanes to reach minimal delay in this situation instead of the 9 in the situation with automatic and electronic lanes.

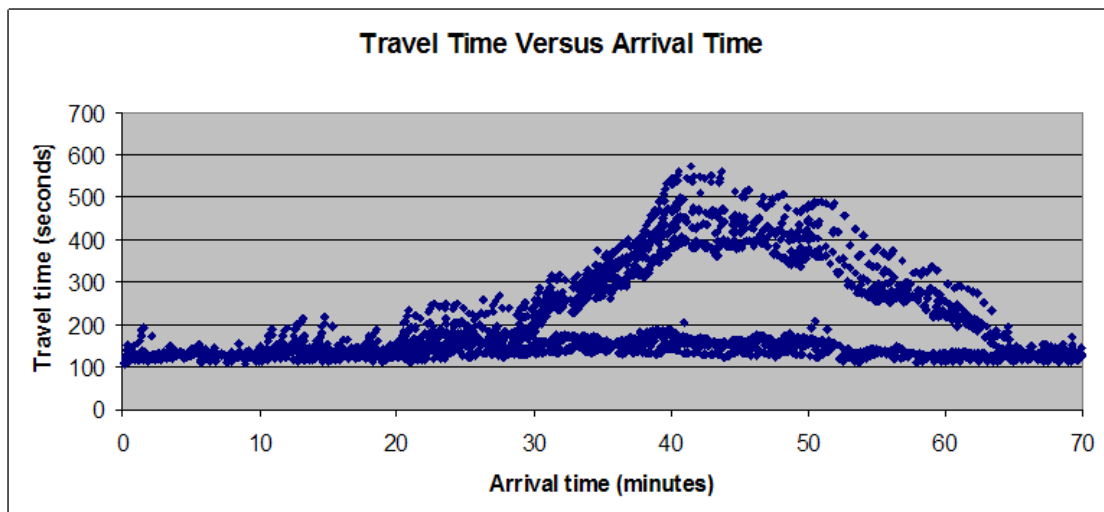
## Do the Cars Behave Reasonably?

In **Figure 6**, we display a graph of travel time vs. arrival time for all cars, under normal traffic loads with no barriers, 4 highway lanes, 8 tollbooths (from left to right: 2 electronic, 4 automatic, 2 manual). There are two main features:

- The line represents cars with electronic passes. Even under a heavy traffic load, they are not terribly delayed, since they can pass through their booth without stopping.
- Along the top, we see the cars without electronic passes. The distribution of their wait times looks like the graph of their generation rate shifted over by about 10 min. One can even see the graph split into several “lanes,” which shows the difference between the slower truck lanes (manual tollbooths) and the normal cars (automatic tollbooths).

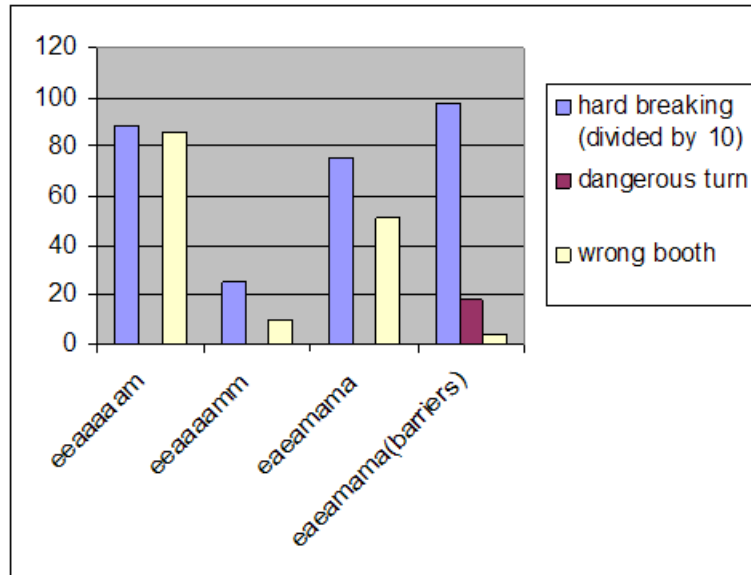


**Figure 5.** Delay vs. number of tollbooths—no trucks or electronic passes.



**Figure 6.** Travel time vs. arrival time.

We are also interested in what configurations lead to potential accidents. We ran setups under the default parameters of normal traffic load, 50% electronic passes, and 10% trucks. **Figure 7** shows the number of occurrences of several types of these behaviors, out of about 3000 cars total.



**Figure 7.** Incidents of dangerous behavior.

We see in the leftmost configuration that having only one manual lane leads to trucks stuck in the wrong booth. Trucks joining the wrong booth also seems to lead to an increase in hard braking; this appears to be an artifact of cars traveling at speeds of 1 or 2 not decelerating properly when nearing a tollbooth. In our experiments, this phenomenon tends to be correlated with inefficient lane-changing schemes. The second configuration from the left is our recommended configuration. The third and fourth configurations show the difference that barriers make: They cause fewer tollbooth mistakes but lead to dangerous turns and hard braking, which are probably related.

## Sensitivity to Parameters

Changing the length of the lane-expansion and -contraction regions did not have a statistically significant effect on either wait times or logs of bad behaviors. The percentages of cars with electronic passes and trucks can be changed by a fair amount before they affect anything. For a general number  $n$  of highway lanes and rigid barriers, the marginal return of adding a new tollbooth after  $2n$  or  $2n + 1$  is small unless the traffic load is exceptionally large.

## Strengths and Weaknesses

### Strengths

**Can handle a wide variety of possible setups.** It is hard to add new tollbooths but easier to change the type of tollbooth or set up barriers.

**Captures important features of actual situation.**

**Behavior based on simple procedures meant to accomplish natural goals.** We avoid introducing artificial effects by basing drivers' behaviors on simple methods of accomplishing natural goals, such as avoiding collisions and getting into a better lane.

### Weaknesses

**Need to obtain real-world parameters.** If we were acting as consultants for a particular highway, we should collect data.

**More complicated than simple models in literature.** Our model may introduce some artificial behavior. Cellular automaton models are supposed to have complex behavior emerge from simple assumptions, not the other way around.

**Infinite speed of information propagation.** Due to the order in which cars are updated in our model, information about obstacles can propagate backwards at infinite speed, an effect which can lead to inaccuracies.

## Conclusion

Cellular-automaton models are one effective means of studying traffic simulations. Other approaches use partial differential equations motivated by kinetics or fluid mechanics [Chowdhury et al. 1997, 213–225].

Our cellular automaton model gives us valuable insight into the tollbooth traffic problem. We can see cars flowing through the tollbooths and piling up during rush hour. We can follow the motions of individual cars and collect statistics on their behaviors. From our experiments, we make the following recommendations:

**Tollbooths should be ordered based on encouraged behavior.** Safety considerations should take precedence; putting faster booths on the left and slower booths on the right accomplishes this.

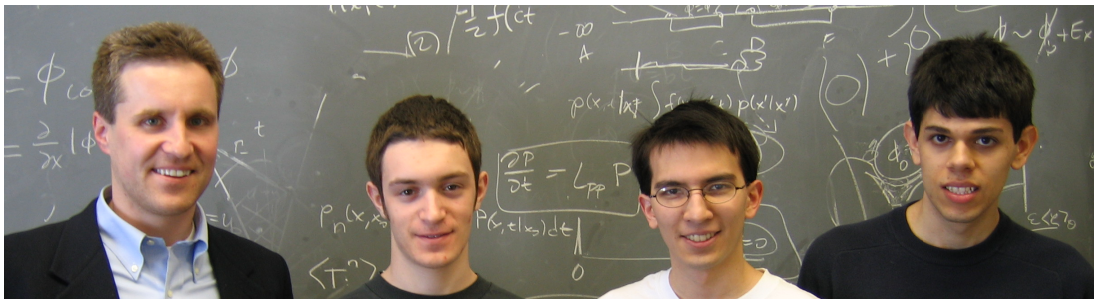
**No barriers.** Barriers prevent drivers getting to lanes that they need to use.

**The distribution of types of cars should determine how many tollbooths.** Traffic density has little effect on the number of tollbooths needed to minimize delay; the distribution of types of cars has a much larger effect.

**An effective ratio of tollbooths is 1 electronic: 2 automatic : 1 manual.**

## References

- Chowdhury, Debashish, Ludger Santen, and Andreas Schadschneider. 2000. Statistical physics of vehicular traffic and some related systems. *Physics Reports* 329: 199–329.
- Chowdhury, Debashish, Dietrich E. Wolf, and Michael Schreckenberg. 1997. Particle hopping models for two-lane traffic with two kinds of vehicles: Effects of lane-changing rules. *Physica A* 235: 417–439.
- Haberman, Richard. 1977. *Mathematical Models: Mechanical Vibrations, Population Dynamics, and Traffic Flow*. Englewood Cliffs, NJ: Prentice Hall. 1998. Reprint. Philadelphia: SIAM.
- Hall, Fred L, Brian L. Allen, and Margot A. Gunter. 1986. Empirical analysis of freeway flow-density relationships. *Transportation Research Part A: General* 20 (3): 197–210.
- Huang, Ding-wei, and Wei-neng Huang. 2002. The influence of tollbooths on highway traffic. *Physica A* 312: 597–608.
- Klodzinski, Jack, and Haitham Al-Deek. 2002. New methodology for defining level of service at toll plazas. *Journal of Transportation Engineering* 128 (2): 173–181.
- Nagel, Kai, and Michael Schreckenberg. 1992. A cellular automaton model for freeway traffic. *Journal de Physique I* 2 (12): 2221–2229.
- Rickert, M., K. Nagel, M. Schreckenberg, and A. Latour. 1996. Two lane traffic simulations using cellular automata. *Physica A* 231: 534–550.



Prof. Martin Bazant, Daniel Kane, Andrew Spann, and Daniel Gulotta.